

GENERALIZED COINVARIANT ALGEBRAS FOR WREATH PRODUCTS

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ABSTRACT. Let r be a positive integer and let G_n be the reflection group of $n \times n$ monomial matrices whose entries are r^{th} complex roots of unity and let $k \leq n$. We define and study two new graded quotients $R_{n,k}$ and $S_{n,k}$ of the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$ in n variables. When $k = n$, both of these quotients coincide with the classical coinvariant algebra attached to G_n . The algebraic properties of our quotients are governed by the combinatorial properties of k -dimensional faces in the Coxeter complex attached to G_n (in the case of $R_{n,k}$) and r -colored ordered set partitions of $\{1, 2, \dots, n\}$ with k blocks (in the case of $S_{n,k}$). Our work generalizes a construction of Haglund, Rhoades, and Shimozono from the symmetric group \mathfrak{S}_n to the more general wreath products G_n .

1. INTRODUCTION

The coinvariant algebra of the symmetric group \mathfrak{S}_n is among the most important \mathfrak{S}_n -modules in combinatorics. It is a graded version of the regular representation of \mathfrak{S}_n , has structural properties deeply tied to the combinatorics of permutations, and gives a combinatorially accessible model for the action of \mathfrak{S}_n on the cohomology ring $H^\bullet(G/B)$ of the flag manifold G/B .

Haglund, Rhoades, and Shimozono [14] recently defined a generalization of the \mathfrak{S}_n -coinvariant algebra which depends on an integer parameter $k \leq n$. The structure of their graded \mathfrak{S}_n -module is governed by the combinatorics of ordered set partitions of $[n] := \{1, 2, \dots, n\}$ with k blocks. The graded Frobenius images of this module is (up to a minor twist) either of the combinatorial expressions $\text{Rise}_{n,k}(\mathbf{x}; q, t)$ or $\text{Val}_{n,k}(\mathbf{x}; q, t)$ appearing in the *Delta Conjecture* of Haglund, Remmel, and Wilson [13] upon setting $t = 0$. The Delta Conjecture is a generalization of the Shuffle Conjecture in the field of Macdonald polynomials; this gives the first example of a ‘naturally constructed’ module with Frobenius image related to the Delta Conjecture.

A linear transformation $t \in GL_n(\mathbb{C})$ is a *reflection* if the fixed space of t has codimension 1 in \mathbb{C}^n and t has finite order. A finite subgroup $W \subseteq GL_n(\mathbb{C})$ is called a *reflection group* if W is generated by reflections. Given any complex reflection group W , there is a coinvariant algebra R_W attached to W . The algebra R_W is a graded W -module with structural properties closely related to the combinatorics of W . In this paper we provide a Haglund-Rhoades-Shimozono style generalization of R_W in the case where R_W belongs to the family of reflection groups $G(r, 1, n) = \mathbb{Z}_r \wr \mathfrak{S}_n$.

The general linear group $GL_n(\mathbb{C})$ acts on the polynomial ring $\mathbb{C}[\mathbf{x}_n] := \mathbb{C}[x_1, \dots, x_n]$ by linear substitutions. If $W \subset GL_n(\mathbb{C})$ is any finite subgroup, let

$$\mathbb{C}[\mathbf{x}_n]^W := \{f(\mathbf{x}_n) \in \mathbb{C}[\mathbf{x}_n] : w.f(\mathbf{x}_n) = f(\mathbf{x}_n) \text{ for all } w \in W\}$$

denote the associated subspace of *invariant polynomials* and let $\mathbb{C}[\mathbf{x}_n]_+^W \subset \mathbb{C}[\mathbf{x}_n]^W$ denote the collection of invariant polynomials with vanishing constant term. The *invariant ideal* $I_W \subset \mathbb{C}[\mathbf{x}_n]$ is the ideal $I_W := \langle \mathbb{C}[\mathbf{x}_n]_+^W \rangle$ generated by $\mathbb{C}[\mathbf{x}_n]_+^W$ and the *coinvariant algebra* is $R_W := \mathbb{C}[\mathbf{x}_n]/I_W$. The quotient R_W is a graded W -module. A celebrated result of Chevalley [6] states that if W is a complex reflection group, then R_W is isomorphic to the regular representation $\mathbb{C}[W]$ as a W -module.

Key words and phrases. Coxeter complex, coinvariant algebra, wreath product.

Notation. Throughout this paper r will denote a positive integer. Unless otherwise stated, we assume $r \geq 2$. Let $\zeta := e^{\frac{2\pi i}{r}} \in \mathbb{C}$ and let $G := \langle \zeta \rangle$ be the multiplicative group of r^{th} roots of unity in \mathbb{C}^\times .

Let us introduce the family of reflection groups we will focus on. A matrix is *monomial* if it has a unique nonzero entry in every row and column. Let G_n be the group of $n \times n$ monomial matrices whose nonzero entries lie in G . For example, if $r = 3$ we have

$$g = \begin{pmatrix} 0 & 0 & \zeta & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \zeta^2 \\ \zeta & 0 & 0 & 0 \end{pmatrix} \in G_4.$$

Matrices in G_n may be thought of combinatorially as r -colored permutations $\pi_1^{c_1} \dots \pi_n^{c_n}$, where $\pi_1 \dots \pi_n$ is a permutation in \mathfrak{S}_n and $c_1 \dots c_n$ is a sequence of ‘colors’ in the set $\{0, 1, \dots, r-1\}$ representing powers of ζ . For example, the above element of G_4 may be represented combinatorially as $g = 4^1 2^0 1^1 3^2$.

In the usual classification of complex reflection groups we have $G_n = G(r, 1, n)$. The group G_n is isomorphic to the wreath product $\mathbb{Z}_r \wr \mathfrak{S}_n = (\mathbb{Z}_r \times \dots \times \mathbb{Z}_r) \rtimes \mathfrak{S}_n$, where the symmetric group \mathfrak{S}_n acts on the n -fold direct product of cyclic groups $\mathbb{Z}_r \times \dots \times \mathbb{Z}_r$ by coordinate permutation. For the sake of legibility, we suppress reference to r in our notation for G_n and related objects.

Let $I_n \subseteq \mathbb{C}[\mathbf{x}_n]$ be the invariant ideal associated to G_n . We have $I_n = \langle e_1(\mathbf{x}_n^r), \dots, e_n(\mathbf{x}_n^r) \rangle$, where

$$e_d(\mathbf{x}_n^r) = e_d(x_1^r, \dots, x_n^r) := \sum_{1 \leq i_1 < \dots < i_d \leq n} x_{i_1}^r \cdots x_{i_d}^r$$

is the d^{th} elementary symmetric function in the variable powers x_1^r, \dots, x_n^r . Let $R_n := \mathbb{C}[\mathbf{x}_n]/I_n$ denote the coinvariant ring attached to G_n .

The algebraic properties of the quotient R_n are governed by the combinatorial properties of r -colored permutations in G_n . Chevalley’s result [6] implies that $R_n \cong \mathbb{C}[G_n]$ as ungraded G_n -modules. The fact that $e_1(\mathbf{x}_n^r), \dots, e_n(\mathbf{x}_n^r)$ is a regular sequence in $\mathbb{C}[\mathbf{x}_n]$ gives the following expression for the Hilbert series of R_n :

$$(1.1) \quad \text{Hilb}(R_n; q) = \prod_{i=1}^n \frac{1 - q^{ri}}{1 - q} = \sum_{g \in G_n} q^{\text{maj}(g)},$$

where maj is the *major index* statistic on G_n (also known as the *flag-major index*; see [12]). Bango and Biagoli [4] described a *descent monomial basis* $\{b_g : g \in G_n\}$ of R_n whose elements satisfy $\deg(b_g) = \text{maj}(b_g)$. Stembridge [22, Thm. 6.6] described the graded G_n -module structure of R_n using (the $r \geq 1$ generalization of) standard Young tableaux.

When $r = 1$ and $G_n = \mathfrak{S}_n$ is the symmetric group, Haglund, Rhoades, and Shimozono [14, Defn. 1.1] introduced and studied a generalization of the coinvariant algebra R_n depending on a positive integer $k \leq n$. In this paper we extend [14, Defn. 1.1] to $r \geq 2$ by introducing the following two families of ideals $I_{n,k}, J_{n,k} \subseteq \mathbb{C}[\mathbf{x}_n]$.

Definition 1.1. Let n, k , and r be nonnegative integers which satisfy $n \geq k, n \geq 1$, and $r \geq 2$. We define two quotients of the polynomial ring $\mathbb{C}[\mathbf{x}_n]$ as follows.

- (1) Let $I_{n,k} \subseteq \mathbb{C}[\mathbf{x}_n]$ be the ideal

$$I_{n,k} := \langle x_1^{kr+1}, x_2^{kr+1}, \dots, x_n^{kr+1}, e_n(\mathbf{x}_n^r), e_{n-1}(\mathbf{x}_n^r), \dots, e_{n-k+1}(\mathbf{x}_n^r) \rangle$$

and let $R_{n,k}$ be the corresponding quotient:

$$R_{n,k} := \mathbb{C}[\mathbf{x}_n]/I_{n,k}.$$

(2) Let $J_{n,k} \subseteq \mathbb{C}[\mathbf{x}_n]$ be the ideal

$$J_{n,k} := \langle x_1^{kr}, x_2^{kr}, \dots, x_n^{kr}, e_n(\mathbf{x}_n^r), e_{n-1}(\mathbf{x}_n^r), \dots, e_{n-k+1}(\mathbf{x}_n^r) \rangle$$

and let $S_{n,k}$ be the corresponding quotient:

$$S_{n,k} := \mathbb{C}[\mathbf{x}_n] / J_{n,k}.$$

Both of the ideals $I_{n,k}$ and $J_{n,k}$ are homogeneous and stable under the action of G_n on $\mathbb{C}[\mathbf{x}_n]$. It follows that the quotients $R_{n,k}$ and $S_{n,k}$ are graded G_n -modules. The ring introduced in [14, Defn. 1.1] is the ideal $S_{n,k}$ with $r = 1$.

When $k = n$, it can be shown¹ that for any $1 \leq i \leq n$, the variable power x_i^{nr} lies in the invariant ideal I_n , so that $I_{n,n} = J_{n,n} = I_n$, and $R_{n,n} = S_{n,n}$ are both equal to the classical coinvariant algebra R_n for G_n . At the other extreme, we have $R_{n,0} \cong \mathbb{C}$ (the trivial representation in degree 0) and $S_{n,0} = 0$.

The reader may wonder why we are presenting two generalizations of the ring of [14] rather than one. The combinatorial reason for this is the presence of *zero blocks* in the G_n -analog of ordered set partitions. These zero blocks do not appear in the case of [14] when $r = 1$ (or in the case of the classical coinvariant algebra when $k = n$). Roughly speaking, the ring $S_{n,k}$ will be a ‘zero block free’ version of $R_{n,k}$. These rings will be related in a nice way (see Proposition 6.1), and $S_{n,k}$ will be easier to analyze directly.

The generators of the ideal $I_{n,k}$ defining the quotient $R_{n,k}$ come in two flavors:

- high degree invariant polynomials $e_n(\mathbf{x}_n^r), e_{n-1}(\mathbf{x}_n^r), \dots, e_{n-k+1}(\mathbf{x}_n^r)$, and
- a collection of polynomials $x_1^{kr+1}, \dots, x_n^{kr+1}$ whose linear span $\text{span}\{x_1^{kr+1}, \dots, x_n^{kr+1}\}$ is stable under the action of G_n and carries the dual of the defining action of G_n on \mathbb{C}^n .

This extends the two flavors of generators for the ideal of [14]. In the context of the 0-Hecke algebra $H_n(0)$ attached to the symmetric group, Huang and Rhoades [15] defined another ideal (denoted in [15] by $J_{n,k} \subseteq \mathbb{F}[\mathbf{x}_n]$, where \mathbb{F} is any field) with analogous types of generators: high degree $H_n(0)$ -invariants together with a copy of the defining representation of $H_n(0)$ sitting in homogeneous degree k . It would be interesting to see if the favorable properties of the corresponding quotients could be derived from this choice of generator selection in a more conceptual way.

In this paper we will prove that the structures of the rings $R_{n,k}$ and $S_{n,k}$ are controlled by G_n -generalizations of ordered set partitions. We will use the usual q -analog notation

$$\begin{aligned} [n]_q &:= 1 + q + \dots + q^{n-1} & [n]!_q &:= [n]_q [n-1]_q \cdots [1]_q \\ \left[\begin{matrix} n \\ a_1, \dots, a_r \end{matrix} \right]_q &:= \frac{[n]!_q}{[a_1]!_q \cdots [a_r]!_q} & \left[\begin{matrix} n \\ a \end{matrix} \right]_q &:= \frac{[n]!_q}{[a]!_q [n-a]!_q}. \end{aligned}$$

We also let rev_q be the operator which reverses the coefficient sequences in polynomials in the variable q (over any ground ring). For example, we have

$$\text{rev}_q(8q^2 + 7q + 6) = 6q^2 + 7q + 8.$$

Let $\text{Stir}(n, k)$ be the (signless) Stirling number of the second kind counting set partitions of $[n]$ into k blocks and let $\text{Stir}_q(n, k)$ denote the q -Stirling number defined by the recursion

$$\text{Stir}_q(n, k) = [k]_q \cdot \text{Stir}_q(n-1, k) + \text{Stir}_q(n-1, k-1)$$

for $n, k \geq 1$ and the initial condition $\text{Stir}_q(0, k) = \delta_{0,k}$. Deferring various definitions to Section 2, we state our main results.

- As *ungraded* G_n -modules we have

$$R_{n,k} \cong \mathbb{C}[\mathcal{F}_{n,k}] \text{ and } S_{n,k} \cong \mathbb{C}[\mathcal{OP}_{n,k}],$$

¹By [5, Sec. 7.2] under the change of variables $(x_1, \dots, x_n) \mapsto (x_1^r, \dots, x_n^r)$ we have $x_n^{nr} \in I_n$, and the ideal I_n is stable under \mathfrak{S}_n .

where $\mathcal{F}_{n,k}$ is the set of k -dimensional faces in the Coxeter complex attached to G_n and $\mathcal{OP}_{n,k}$ is the set of r -colored ordered set partitions of $[n]$ with k blocks (Corollary 4.12). In particular, we have

$$\begin{aligned}\dim(R_{n,k}) &= \sum_{z=0}^{n-k} \binom{n}{z} \cdot r^{n-z} \cdot k! \cdot \text{Stir}(n-z, k), \\ \dim(S_{n,k}) &= r^n \cdot k! \cdot \text{Stir}(n, k).\end{aligned}$$

- The Hilbert series $\text{Hilb}(R_{n,k}; q)$ and $\text{Hilb}(S_{n,k}; q)$ are given by (Corollary 4.11)

$$\begin{aligned}\text{Hilb}(R_{n,k}; q) &= \sum_{z=0}^{n-k} \binom{n}{z} \cdot q^{krz} \cdot \text{rev}_q([r]_q^{n-z} \cdot [k]_{q^r}! \cdot \text{Stir}_{q^r}(n-z, k)), \\ \text{Hilb}(S_{n,k}; q) &= \text{rev}_q([r]_q^n \cdot [k]_{q^r}! \cdot \text{Stir}_{q^r}(n, k)).\end{aligned}$$

- Endow monomials in $\mathbb{C}[\mathbf{x}_n]$ with the lexicographic term order. The standard monomial basis of $R_{n,k}$ is the collection of monomials $m = x_1^{a_1} \cdots x_n^{a_n}$ whose exponent sequences (a_1, \dots, a_n) are componentwise \leq some shuffle of the sequences $(r-1, 2r-1, \dots, kr-1)$ and $\underbrace{(kr, \dots, kr)}_{n-k}$.

The standard monomials basis of $S_{n,k}$ is the collection of monomials $m = x_1^{b_1} \cdots x_n^{b_n}$ whose exponent sequences (b_1, \dots, b_n) are componentwise \leq some shuffle of the sequences $(r-1, 2r-1, \dots, kr-1)$ and $\underbrace{(kr-1, \dots, kr-1)}_{n-k}$ (Theorem 4.13).

- There is a generalization of Bango and Biagoli's descent monomial basis of R_n to the rings $R_{n,k}$ and $S_{n,k}$ (Theorems 5.8 and 5.10).
- We have an explicit description of the *graded* isomorphism type of the G_n -modules $R_{n,k}$ and $S_{n,k}$ in terms of standard Young tableaux (Theorem 6.14).

Although the properties of the rings $R_{n,k}$ (and $S_{n,k}$) shown above give natural extensions of the corresponding properties of R_n , the proofs of these results will be quite different. Since the classical invariant ideal I_n is cut out by a regular sequence $e_1(\mathbf{x}_n^r), \dots, e_n(\mathbf{x}_n^r)$, standard tools from commutative algebra (the *Koszul complex*) can be used to derive the graded isomorphism type of R_n . Since neither the dimension $\dim(R_{n,k}) = \sum_{z=0}^{n-k} \binom{n}{z} \cdot r^{n-z} \cdot k! \cdot \text{Stir}(n-z, k)$ nor $\dim(S_{n,k}) = r^n \cdot k! \cdot \text{Stir}(n, k)$ have nice product formulas, we cannot hope to apply this technology to our situation.

Replacing the commutative algebra machinery used to analyze R_n will be *combinatorial* commutative algebra machinery (Gröbner theory and straightening laws) which will determine the structure of $R_{n,k}$. Although some portions of our analysis will follow relatively easily from the arguments of [14] after making the change of variables $(x_1, \dots, x_n) \mapsto (x_1^r, \dots, x_n^r)$, other arguments will have to be significantly adapted to account for the possible presence of zero blocks.

The rest of the paper is organized as follows. In **Section 2** we give background material related to r -colored ordered set partitions, the Coxeter complex of G_n , symmetric functions, the representation theory of G_n , and Gröbner theory. In **Section 3** we prove some polynomial and symmetric function identities that will be helpful in later sections. In **Section 4** we calculate the standard monomial bases of $R_{n,k}$ and $S_{n,k}$ with respect to the lexicographic term order and calculate the Hilbert series of these quotients. In **Section 5** we present our generalizations of the Bango-Biagoli descent monomial basis of R_n to obtain descent monomial-type bases for $R_{n,k}$ and $S_{n,k}$. In **Section 6** we derive the graded isomorphism type of the G_n -modules $R_{n,k}$ and $S_{n,k}$. We close in **Section 7** with some open questions.

2. BACKGROUND

2.1. **r -colored ordered set partitions.** We will make use of two orders on the alphabet

$$\mathcal{A}_r := \{i^c : i \in \mathbb{Z}_{>0} \text{ and } 0 \leq c \leq r-1\}$$

of r -colored positive integers. The first order $<$ weights colors more heavily than letter values, with higher colors being smaller:

$$1^{r-1} < 2^{r-1} < \dots < 1^{r-2} < 2^{r-2} < \dots < 1^0 < 2^0 < \dots.$$

The second order \prec weights letter values more heavily than colors:

$$1^{r-1} \prec 1^{r-2} \prec \dots \prec 1^0 \prec 2^{r-1} \prec 2^{r-2} \prec \dots \prec 2^0 \prec \dots.$$

Let $w = w_1^{c_1} \dots w_n^{c_n}$ be any word in the alphabet \mathcal{A}_r . The *descent set* and *ascent set* of w are defined using the order $<$:

$$(2.1) \quad \text{Des}(w) := \{1 \leq i \leq n-1 : w_i^{c_i} > w_{i+1}^{c_{i+1}}\}, \quad \text{Asc}(w) := \{1 \leq i \leq n-1 : w_i^{c_i} < w_{i+1}^{c_{i+1}}\}.$$

We write $\text{des}(w) := |\text{Des}(w)|$ and $\text{asc}(w) := |\text{Asc}(w)|$ for the number of descents and ascents in w . The *major index* $\text{maj}(w)$ is given by the formula

$$(2.2) \quad \text{maj}(w) := c(w) + r \cdot \sum_{i \in \text{Des}(w)} i,$$

where $c(w)$ denotes the sum of the colors of the letters in w . This version of major index was defined by Haglund, Loehr, and Remmel in [12] (where it was termed ‘flag-major index’).

Since we may view elements of G_n as r -colored permutations, the objects defined in the above paragraph make sense for $g \in G_n$. For example, if $r = 3$ and $g = 3^0 4^1 6^2 2^0 5^2 1^2 \in G_6$, we have $\text{Des}(g) = \{1, 2, 4, 5\}$, $\text{Asc}(g) = \{3\}$, $\text{des}(g) = 4$, $\text{asc}(g) = 1$, and

$$\text{maj}(g) = (0 + 1 + 2 + 0 + 2 + 2) + 3 \cdot (1 + 2 + 4 + 5) = 43.$$

An *ordered set partition* is a set partition equipped with a total order on its blocks. An *r -colored ordered set partition of size n* is an ordered set partition σ of $[n]$ in which every letter is assigned a color in the set $\{0, 1, \dots, r-1\}$. For example,

$$\sigma = \{3^0, 4^1\} \prec \{6^2\} \prec \{1^2, 2^0, 5^0\}$$

is a 3-colored ordered set partition of size 6 with 3 blocks. We let $\mathcal{OP}_{n,k}$ be the collection of r -colored ordered set partitions of size n with k blocks. We have

$$(2.3) \quad |\mathcal{OP}_{n,k}| = r^n \cdot k! \cdot \text{Stir}(n, k).$$

We will often use bars to represent colored ordered set partitions more succinctly. Here we write block elements in increasing order with respect to \prec . Our example ordered set partition becomes

$$\sigma = (3^0 4^1 \mid 6^2 \mid 1^2 2^0 5^2).$$

We also have a descent starred notation for colored ordered set partitions, where we order elements within blocks in a decreasing fashion with respect to $<$. Our example ordered set partition becomes

$$\sigma = 3_*^0 4_*^1 \ 6^2 \ 2_*^0 5_*^2 1_*^2.$$

Notice that we use the order \prec for the bar notation, but the order $<$ for the star notation. The star notation represents $\sigma \in \mathcal{OP}_{n,k}$ as a pair $\sigma = (g, S)$, where $g \in G_n$, $|S| = n - k$ and $S \subseteq \text{Des}(g)$. Our example ordered set partition becomes

$$\sigma = (3^0 4^1 6^2 2^0 5^2 1^2, \{1, 4, 5\}).$$

Let $\sigma \in \mathcal{OP}_{n,k}$ and let (g, S) be the descent starred representation of σ . The *major index* of $\sigma = (g, S)$ is

$$(2.4) \quad \text{maj}(\sigma) = \text{maj}(g, S) = c(\sigma) + r \cdot \left[\sum_{i \in \text{Des}(g)} i - \sum_{i \in S} |\text{Des}(g) \cap \{i, i+1, \dots, n\}| \right],$$

where $c(\sigma)$ denotes the sum of the colors in σ . In the example above, we have

$$\text{maj}(3^0_4 4^1 \mid 6^2 \mid 2^0_5 2^2_1) = (0 + 1 + 2 + 0 + 2 + 2) + 3 \cdot [(1 + 2 + 4 + 5) - (4 + 2 + 1)] = 22.$$

Whereas the definition of maj for colored ordered set partitions used the order $<$ to compare elements, the definition of coinv uses the order \prec . In particular, let σ be a colored ordered set partition. A *coinversion pair* in σ is a pair of colored letters $i^c \preceq j^d$ appearing in σ such that

$$\begin{cases} \text{at least one of } i^c \text{ and } j^d \text{ is } \prec\text{-minimal in its block in } \sigma, \\ i^c \text{ and } j^d \text{ belong to different blocks of } \sigma, \text{ and} \\ \text{if } i^c\text{'s block is to the right of } j^d\text{'s block, then only } j^d \text{ is } \prec\text{-minimal in its block.} \end{cases}$$

In our example $\sigma = (3^0_4 4^1 \mid 6^2 \mid 1^2 2^0 5^2)$, the coinversion pairs are $3^0 6^2, 2^0 3^0, 3^0 5^2, 2^0 6^2, 4^1 6^2$, and $5^2 6^2$. The statistic $\text{coinv}(\sigma)$ is defined by

$$(2.5) \quad \text{coinv}(\sigma) = [n \cdot (r-1) - c(\sigma)] + r \cdot (\text{number of coinversion pairs in } \sigma).$$

In our example we have

$$\text{coinv}(3^0_4 4^1 \mid 6^2 \mid 1^2 2^0 5^2) = [6 \cdot 2 - (0 + 1 + 2 + 2 + 0 + 2)] + 3 \cdot 6 = 23.$$

In particular, whereas the statistic maj involves a sum over colors, the statistic coinv involves a sum over *complements* of colors. The statistic coinv on r -colored k -block ordered set partitions of $[n]$ is complementary to the statistic inv defined in [18, Sec. 4].

We need an extension of colored set partitions involving repeated letters. An *r -colored ordered multiset partition* μ is a sequence of finite nonempty sets $\mu = (M_1, \dots, M_k)$ of elements from the alphabet \mathcal{A}_r . The *size* of μ is $|M_1| + \dots + |M_k|$ and we say that μ has k *blocks*. For example, we have that $\mu = (2^1 1^2 3^1 \mid 1^2 3^1 \mid 2^0 4^2)$ is a 3-colored ordered multiset partition of size 7 with 3 blocks.

We emphasize that the blocks of ordered multiset partitions are *sets*; there are no repeated letters within blocks (although the same letter can occur with different colors within a single block). If μ is an ordered multiset partition, the statistics $\text{coinv}(\mu)$ and $\text{maj}(\mu)$ have the same definitions as in the case of no repeated letters.

2.2. G_n -faces. To describe the combinatorics of the rings $R_{n,k}$, we introduce the following concept of a G_n -face. In the following definition we require $r \geq 2$.

Definition 2.1. A G_n -face is an ordered set partition $\sigma = (B_1 \mid B_2 \mid \dots \mid B_m)$ of $[n]$ such that the letters in every block of σ , with the possible exception of the first block B_1 , are decorated by the colors $\{0, 1, \dots, r-1\}$.

Let $\sigma = (B_1 \mid B_2 \mid \dots \mid B_m)$ be an G_n -face. If the letters in B_1 are uncolored, then B_1 is called the *zero block* of σ . The *dimension* of σ is the number of nonzero blocks in σ . Let $\mathcal{F}_{n,k}$ denote the set of G_n -faces of dimension k . For example, if $r = 3$ we have

$$\begin{aligned} (25 \mid 1^1 3^2 6^2 \mid 4^1) &\in \mathcal{F}_{6,2} \text{ and} \\ (2^2 5^1 \mid 1^1 3^2 6^2 \mid 4^1) &\in \mathcal{F}_{6,3}, \end{aligned}$$

where the lack of colors on the letters of the first block $\{2, 5\}$ of the top face indicates that $\{2, 5\}$ is a zero block. When $k = n$, we have $\mathcal{F}_{n,n} = \mathcal{OP}_{n,n} = G_n$ as there cannot be a zero block.

The notation *face* in Definition 2.1 comes from the identification of the k -dimensional G_n -faces with the k -dimensional faces in the Coxeter complex of G_n . The set $\mathcal{F}_{n,k}$ may also be identified

with the collection of rank k elements in the Dowling lattice $Q_n(\Gamma)$ to a group Γ of size r (see [7]). By considering the possible sizes of zero blocks, we see that the number of faces in $\mathcal{F}_{n,k}$ is

$$(2.6) \quad |\mathcal{F}_{n,k}| = \sum_{z=0}^{n-k} \binom{n}{z} \cdot r^{n-z} \cdot k! \cdot \text{Stir}(n-z, k).$$

We will consider an action of the group G_n on $\mathcal{F}_{n,k}$. To describe this action it suffices to describe the action of permutation matrices $\mathfrak{S}_n \subseteq G_n$ and the diagonal subgroup $\mathbb{Z}_r \times \cdots \times \mathbb{Z}_r \subseteq G_n$. If $\pi = \pi_1 \dots \pi_n \in \mathfrak{S}_n$, then π acts on G_n by swapping letters while preserving colors. For example, if $\pi = 614253 \in \mathfrak{S}_6$, then

$$\pi.(25 \mid 1^1 3^2 6^2 \mid 4^1) = (15 \mid 6^1 4^2 3^2 \mid 2^1) = (15 \mid 3^2 4^2 6^1 \mid 2^1).$$

A diagonal matrix $g = \text{diag}(\zeta^{c_1}, \dots, \zeta^{c_n})$ acts by increasing the color of the letter i by $c_i \pmod{r}$, while leaving elements in the zero block uncolored. For example, if $r = 3$ an example action of the diagonal matrix $g = \text{diag}(\zeta, \zeta^2, \zeta^2, \zeta, \zeta^2, \zeta) \in G_6$ is

$$g.(25 \mid 1^1 3^2 6^2 \mid 4^1) = (25 \mid 1^2 3^1 6^0 \mid 4^2).$$

It is clear that the action of G_n on $\mathcal{F}_{n,k}$ preserves the subset $\mathcal{OP}_{n,k}$ of r -colored ordered set partitions.

We extend the definition of coinvariant to G_n -faces as follows. There is a natural map

$$(2.7) \quad \pi : \mathcal{F}_{n,k} \rightarrow \bigcup_{z=0}^{n-k} \mathcal{OP}_{n-z,k}$$

which removes the zero block Z of a G_n -face (if present), and then maps the letters in $[n] - Z$ onto $\{1, 2, \dots, n - |Z|\}$ via an order-preserving bijection while preserving colors. For example, we have

$$\pi : (25 \mid 1^1 3^2 6^2 \mid 4^1) \mapsto (1^1 2^2 4^2 \mid 3^1).$$

If σ is a G_n -face whose zero block has size z , we define $\text{coinv}(\sigma)$ by

$$(2.8) \quad \text{coinv}(\sigma) := krz + \text{coinv}(\pi(\sigma)).$$

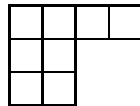
In the $r = 3$ example above, we have

$$\text{coinv}(25 \mid 1^1 3^2 6^2 \mid 4^1) = 2 \cdot 3 \cdot 2 + \text{coinv}(1^1 2^2 4^2 \mid 3^1) = 12 + 8 = 20.$$

2.3. Symmetric functions. For $n \geq 0$, a (weak) composition of n is a sequence $\alpha = (\alpha_1, \dots, \alpha_k)$ of nonnegative integers with $\alpha_1 + \dots + \alpha_k = n$. We write $\alpha \models n$ or $|\alpha| = n$ to indicate that α is a composition of n .

A partition of n is a composition λ of n whose parts are positive and weakly decreasing. We write $\lambda \vdash n$ to indicate that λ is a partition of n . If λ and μ are partitions (of any size) we say that λ dominates μ and write $\lambda \geq_{\text{dom}} \mu$ if $\lambda_1 + \dots + \lambda_i \geq \mu_1 + \dots + \mu_i$ for all $i \geq 1$.

The Ferrers diagram of a partition λ (in English notation) consists of λ_i left-justified boxes in row i . The Ferrers diagram of $(4, 2, 2) \vdash 8$ is shown below. The conjugate λ' of a partition λ is obtained by reflecting the Ferrers diagram across its main diagonal. For example, we have $(4, 2, 2)' = (3, 3, 1, 1)$.



For an infinite sequence of variables $\mathbf{y} = (y_1, y_2, \dots)$, let $\Lambda(\mathbf{y})$ denote the ring of symmetric functions in the variable set \mathbf{y} with coefficients in the field $\mathbb{Q}(q)$. The ring $\Lambda(\mathbf{y}) = \bigoplus_{n \geq 0} \Lambda(\mathbf{y})_n$ is graded by degree. The degree n piece $\Lambda(\mathbf{y})_n$ has vector space dimension equal to the number of partitions of n .

For a partition λ , let

$$m_\lambda(\mathbf{y}), \quad e_\lambda(\mathbf{y}), \quad h_\lambda(\mathbf{y}), \quad s_\lambda(\mathbf{y})$$

be the corresponding *monomial*, *elementary*, (*complete*) *homogeneous*, and *Schur* symmetric functions. As λ varies over the collection of all partitions, these symmetric functions give four different bases for $\Lambda(\mathbf{y})$. Given any composition β whose nonincreasing rearrangement is the partition λ , we extend this notation by setting $e_\beta(\mathbf{y}) := e_\lambda(\mathbf{y})$ and $h_\beta(\mathbf{y}) := h_\lambda(\mathbf{y})$.

Let $\omega : \Lambda(\mathbf{y}) \rightarrow \Lambda(\mathbf{y})$ be the linear map which sends $s_\lambda(\mathbf{y})$ to $s_{\lambda'}(\mathbf{y})$ for all partitions λ . The map ω is an involution and a ring automorphism. For any partition λ , we have $\omega(e_\lambda(\mathbf{y})) = h_\lambda(\mathbf{y})$ and $\omega(h_\lambda(\mathbf{y})) = e_\lambda(\mathbf{y})$.

We let $\langle \cdot, \cdot \rangle$ denote the *Hall inner product* on $\Lambda(\mathbf{y})$. This can be defined by either of the rules $\langle s_\lambda(\mathbf{y}), s_\mu(\mathbf{y}) \rangle = \delta_{\lambda, \mu}$ or $\langle h_\lambda(\mathbf{y}), m_\mu(\mathbf{y}) \rangle = \delta_{\lambda, \mu}$ for all partitions λ, μ . If $F(\mathbf{y}) \in \Lambda(\mathbf{y})$ is any symmetric function, let $F(\mathbf{y})^\perp$ be the linear operator on $\Lambda(\mathbf{y})$ which is adjoint to the operation of multiplication by $F(\mathbf{y})$. That is, we have

$$(2.9) \quad \langle F(\mathbf{y})^\perp G(\mathbf{y}), H(\mathbf{y}) \rangle = \langle G(\mathbf{y}), F(\mathbf{y})H(\mathbf{y}) \rangle$$

for all symmetric functions $G(\mathbf{y}), H(\mathbf{y}) \in \Lambda(\mathbf{y})$.

The representation theory of G_n is analogous to that of \mathfrak{S}_n , but involves r -tuples of objects. Given any r -tuple $\mathbf{o} = (o^{(1)}, o^{(2)}, \dots, o^{(r-1)}, o^{(r)})$ of objects, we define the *dual* \mathbf{o}^* to be the r -tuple

$$(2.10) \quad \mathbf{o}^* := (o^{(r-1)}, \dots, o^{(2)}, o^{(1)}, o^{(r)})$$

obtained by reversing the first $r - 1$ terms in the sequence \mathbf{o} . At the algebraic level, the operator $\mathbf{o} \mapsto \mathbf{o}^*$ corresponds to the entrywise action of complex conjugation on matrices in G_n (which is trivial when $r = 1$ or $r = 2$). If $1 \leq i \leq r$, we define the *dual* i^* of i by the rule

$$(2.11) \quad i^* = \begin{cases} r - i & 1 \leq i \leq r - 1 \\ r & i = r. \end{cases}$$

We therefore have

$$(2.12) \quad \mathbf{o}^* = (o^{(1*)}, \dots, o^{(r*)}) \text{ if } \mathbf{o} = (o^{(1)}, \dots, o^{(r)}).$$

For a positive integer n , an r -*composition* α of n is an r -tuple of compositions $\alpha = (\alpha^{(1)}, \dots, \alpha^{(r)})$ which satisfies $|\alpha| := |\alpha^{(1)}| + \dots + |\alpha^{(r)}| = n$. We write $\alpha \models_r n$ to indicate that α is an r -composition of n .

Similarly, an r -*partition* $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)})$ of n is an r -tuple of partitions with $|\lambda| := |\lambda^{(1)}| + \dots + |\lambda^{(r)}| = n$. We write $\lambda \vdash_r n$ to mean that λ is an r -partition of n . The *conjugate* of an r -partition $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)})$ is defined componentwise; $\lambda' := (\lambda^{(1)'}, \dots, \lambda^{(r)'})$.

The *Ferrers diagram* of an r -partition $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)})$ is the r -tuple of Ferrers diagrams of its constituent partitions. The Ferrers diagram of the 3-partition $((3, 2), \emptyset, (2, 2)) \vdash_3 9$ is shown below.

$$\begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array}, \quad \emptyset, \quad \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline \end{array}$$

Let $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)}) \vdash_r n$ be an r -partition of n . A *semistandard tableau* \mathbf{T} of shape λ is a tuple $\mathbf{T} = (T^{(1)}, \dots, T^{(r)})$, where $T^{(i)}$ is a filling of the boxes of $\lambda^{(i)}$ with positive integers which increase weakly across rows and strictly down columns. A semistandard tableau \mathbf{T} of shape λ is *standard* if the entries $1, 2, \dots, n$ all appear precisely once in \mathbf{T} . Let $\text{SYT}^r(n)$ denote the collection of all possible standard tableaux with r components and n boxes.

For example, let $\lambda = ((3, 2), \emptyset, (2, 2)) \vdash_3 9$. A semistandard tableau $\mathbf{T} = (T^{(1)}, T^{(2)}, T^{(3)})$ of shape λ is

$$\begin{array}{|c|c|c|} \hline 1 & 3 & 3 \\ \hline 3 & 4 & \\ \hline \end{array}, \quad \emptyset, \quad \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 4 & 4 \\ \hline \end{array}.$$

A standard tableau of shape λ is

$$\begin{array}{|c|c|c|} \hline 3 & 6 & 9 \\ \hline 5 & 7 & \\ \hline \end{array}, \quad \emptyset, \quad \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & 8 \\ \hline \end{array}.$$

Let $\mathbf{T} = (T^{(1)}, \dots, T^{(r)}) \in \text{SYT}^r(n)$ be a standard tableau with n boxes. A letter $1 \leq i \leq n-1$ is called a *descent* of \mathbf{T} if

- the letters i and $i+1$ appear in the same component $T^{(j)}$ of \mathbf{T} , and $i+1$ appears in a row below i in $T^{(j)}$, or
- the letter $i+1$ appears in a component of $\mathbf{T} = (T^{(1)}, \dots, T^{(r)})$ strictly to the right of the component containing i .

We let $\text{Des}(\mathbf{T}) := \{1 \leq i \leq n : i \text{ is a descent of } \mathbf{T}\}$ denote the collection of all descents of \mathbf{T} and let $\text{des}(\mathbf{T}) := |\text{Des}(\mathbf{T})|$ denote the number of descents of \mathbf{T} . The *major index* of \mathbf{T} is

$$(2.13) \quad \text{maj}(\mathbf{T}) := r \cdot \sum_{i \in \text{Des}(\mathbf{T})} i + \sum_{j=1}^r (j-1) \cdot |T^{(j)}|,$$

where $|T^{(j)}|$ is the number of boxes in the component $T^{(j)}$. For example, if $\mathbf{T} = (T^{(1)}, T^{(2)}, T^{(3)})$ is the standard tableau above, then $\text{Des}(\mathbf{T}) = \{1, 3, 6, 7\}$, $\text{des}(\mathbf{T}) = 4$, and

$$\text{maj}(\mathbf{T}) = 3 \cdot (1 + 3 + 6 + 7) + (0 \cdot 5 + 1 \cdot 0 + 2 \cdot 4) = 59.$$

For $1 \leq i \leq r$, let $\mathbf{x}^{(i)} = (x_1^{(i)}, x_2^{(i)}, \dots)$ be an infinite list of variables and let $\Lambda(\mathbf{x}^{(i)})$ be the ring of symmetric functions in the variables $\mathbf{x}^{(i)}$ with coefficients in $\mathbb{Q}(q)$. We use \mathbf{x} to denote the union of the r variable sets $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(r)}$. Let $\Lambda^r(\mathbf{x})$ be the tensor product

$$\Lambda^r(\mathbf{x}) = \Lambda(\mathbf{x}^{(1)}) \otimes \dots \otimes \Lambda(\mathbf{x}^{(r)}).$$

We can think of $\Lambda^r(\mathbf{x})$ as the ring of formal power series in $\mathbb{Q}(q)[[\mathbf{x}]]$ which are symmetric in the variable sets $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(r)}$ separately.

The algebra $\Lambda^r(\mathbf{x})$ is spanned by generating tensors of the form

$$F_1(\mathbf{x}^{(1)}) \cdot \dots \cdot F_r(\mathbf{x}^{(r)}) := F_1(\mathbf{x}^{(1)}) \otimes \dots \otimes F_r(\mathbf{x}^{(r)}),$$

where $F_i(\mathbf{x}^{(i)}) \in \Lambda(\mathbf{x}^{(i)})$ is a symmetric function in the variables $\mathbf{x}^{(i)}$. The algebra $\Lambda^r(\mathbf{x})$ is graded via

$$\deg(F_1(\mathbf{x}^{(1)}) \cdot \dots \cdot F_r(\mathbf{x}^{(r)})) := \deg(F_1(\mathbf{x}^{(1)})) + \dots + \deg(F_r(\mathbf{x}^{(r)})),$$

where the $F_i(\mathbf{x}^{(i)})$ are homogeneous.

The standard bases of $\Lambda^r(\mathbf{x})$ are obtained from those of $\Lambda(\mathbf{x}^{(1)}), \dots, \Lambda(\mathbf{x}^{(r)})$ by multiplication. More precisely, let $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$ be an r -partition. We define elements

$$\mathbf{m}_{\boldsymbol{\lambda}}(\mathbf{x}), \mathbf{e}_{\boldsymbol{\lambda}}(\mathbf{x}), \mathbf{h}_{\boldsymbol{\lambda}}(\mathbf{x}), \mathbf{s}_{\boldsymbol{\lambda}}(\mathbf{x}) \in \Lambda^r(\mathbf{x})$$

by

$$\begin{aligned} \mathbf{m}_{\boldsymbol{\lambda}}(\mathbf{x}) &:= m_{\lambda^{(1)}}(\mathbf{x}^{(1)}) \cdots m_{\lambda^{(r)}}(\mathbf{x}^{(r)}), & \mathbf{e}_{\boldsymbol{\lambda}}(\mathbf{x}) &:= e_{\lambda^{(1)}}(\mathbf{x}^{(1)}) \cdots e_{\lambda^{(r)}}(\mathbf{x}^{(r)}), \\ \mathbf{h}_{\boldsymbol{\lambda}}(\mathbf{x}) &:= h_{\lambda^{(1)}}(\mathbf{x}^{(1)}) \cdots h_{\lambda^{(r)}}(\mathbf{x}^{(r)}), & \mathbf{s}_{\boldsymbol{\lambda}}(\mathbf{x}) &:= s_{\lambda^{(1)}}(\mathbf{x}^{(1)}) \cdots s_{\lambda^{(r)}}(\mathbf{x}^{(r)}). \end{aligned}$$

As $\boldsymbol{\lambda}$ varies over the collection of all r -partitions, any of the sets $\{\mathbf{m}_{\boldsymbol{\lambda}}(\mathbf{x})\}, \{\mathbf{e}_{\boldsymbol{\lambda}}(\mathbf{x})\}, \{\mathbf{h}_{\boldsymbol{\lambda}}(\mathbf{x})\}$, or $\{\mathbf{s}_{\boldsymbol{\lambda}}(\mathbf{x})\}$ forms a basis for $\Lambda^r(\mathbf{x})$. If $\boldsymbol{\beta} = (\beta^{(1)}, \dots, \beta^{(r)})$ is an r -composition, we extend this notation by setting

$$\mathbf{e}_{\boldsymbol{\beta}}(\mathbf{x}) := e_{\beta^{(1)}}(\mathbf{x}^{(1)}) \cdots e_{\beta^{(r)}}(\mathbf{x}^{(r)}), \quad \mathbf{h}_{\boldsymbol{\beta}}(\mathbf{x}) := h_{\beta^{(1)}}(\mathbf{x}^{(1)}) \cdots h_{\beta^{(r)}}(\mathbf{x}^{(r)}).$$

The Schur functions $\mathbf{s}_{\boldsymbol{\lambda}}(\mathbf{x})$ admit the following combinatorial description. If $\mathbf{T} = (T^{(1)}, \dots, T^{(r)})$ is a semistandard tableau with r components, let $\mathbf{x}^{\mathbf{T}}$ be the monomial in the variable set \mathbf{x} where the exponent of $x_j^{(i)}$ equals the multiplicity of j in the tableau $T^{(i)}$. For example, if $r = 3$ and $\mathbf{T} = (T^{(1)}, T^{(2)}, T^{(3)})$ is as above, we have

$$\mathbf{x}^{\mathbf{T}} = (x_1^{(1)})^1 (x_3^{(1)})^3 (x_4^{(1)})^1 (x_1^{(3)})^1 (x_3^{(3)})^1 (x_4^{(3)})^2.$$

Similarly, if w is any word in the r -colored positive integers \mathcal{A}_r , let \mathbf{x}^w be the monomial in \mathbf{x} where the exponent of $x_j^{(i)}$ equals the multiplicity of j^{i-1} in the word w . Also, if $\beta = (\beta^{(1)}, \dots, \beta^{(r)})$ is an r -composition, define the monomial \mathbf{x}^β by

$$(2.14) \quad \mathbf{x}^\beta := (x_1^{(1)})^{\beta_1^{(1)}} (x_2^{(1)})^{\beta_2^{(1)}} \cdots (x_1^{(2)})^{\beta_1^{(2)}} (x_2^{(2)})^{\beta_2^{(2)}} \cdots$$

Given an r -partition $\lambda \vdash_r n$, we have

$$(2.15) \quad s_\lambda(\mathbf{x}) = \sum_T \mathbf{x}^T,$$

where the sum is over all semistandard tableaux T of shape λ .

The Hall inner product $\langle \cdot, \cdot \rangle$ extends to $\Lambda^r(\mathbf{x})$ by the rule

$$(2.16) \quad \langle s_\lambda(\mathbf{x}), s_{\mu^*}(\mathbf{x}) \rangle = \langle h_\lambda(\mathbf{x}), m_{\mu^*}(\mathbf{x}) \rangle = \delta_{\lambda, \mu}$$

for all r -partitions λ and μ . The presence of duals in this definition comes from the nontriviality of complex conjugation on G_n for $r > 2$.

The involution ω is defined on $\Lambda^r(\mathbf{x}) = \Lambda(\mathbf{x}^{(1)}) \otimes \cdots \otimes \Lambda(\mathbf{x}^{(r)})$ by applying ω in each component separately. The map ω is an isometry of the inner product $\langle \cdot, \cdot \rangle$.

If $\mathbf{F}(\mathbf{x}) \in \Lambda^r(\mathbf{x})$, we let $\mathbf{F}(\mathbf{x})^\perp$ be the operator on $\Lambda^r(\mathbf{x})$ which is adjoint to multiplication by $\mathbf{F}(\mathbf{x})$ under the inner product $\langle \cdot, \cdot \rangle$. In particular, if $j \geq 1$ and if $1 \leq i \leq r$, we have $h_j(\mathbf{x}^{(i)}), e_j(\mathbf{x}^{(i)}) \in \Lambda^r(\mathbf{x})$, so that $h_j(\mathbf{x}^{(i)})^\perp$ and $e_j(\mathbf{x}^{(i)})^\perp$ make sense as linear operators on $\Lambda^r(\mathbf{x})$. These operators (and their ‘dual’ versions $h_j(\mathbf{x}^{(i^*)})^\perp$ and $e_j(\mathbf{x}^{(i^*)})^\perp$) will play a key role in this paper.

2.4. Representations of G_n . In his thesis, Specht [19] described the irreducible representations of G_n . We recall his construction.

Given a matrix $g \in G_n$, define numbers $\chi(g)$ and $\text{sign}(g)$ by

$$(2.17) \quad \chi(g) := \text{product of the nonzero entries in } g,$$

$$(2.18) \quad \text{sign}(g) := \text{determinant of the permutation matrix underlying } g.$$

In particular, the number $\chi(g)$ is an r^{th} root of unity and $\text{sign}(g) = \pm 1$. Both of the functions χ and sign are linear characters of G_n . In other words, we have $\chi(gh) = \chi(g)\chi(h)$ and $\text{sign}(gh) = \text{sign}(g)\text{sign}(h)$ for all $g, h \in G_n$.

It is well known that the irreducible complex representations of the symmetric group \mathfrak{S}_n are indexed by partitions $\lambda \vdash n$. Given $\lambda \vdash n$, let S^λ be the corresponding irreducible \mathfrak{S}_n -module. For example, we have that $S^{(n)}$ is the trivial representation of \mathfrak{S}_n and $S^{(1^n)}$ is the sign representation of \mathfrak{S}_n .

Let V be a G -module and let U be an \mathfrak{S}_n -module. We build a G_n -module $V \wr U$ by letting $V \wr U = (V)^{\otimes n} \otimes U$ as a vector space and defining the action of G_n by

$$(2.19) \quad \text{diag}(g_1, \dots, g_n) \cdot (v_1 \otimes \cdots \otimes v_n \otimes u) := (g_1.v_1) \otimes \cdots \otimes (g_n.v_n) \otimes u,$$

for all diagonal matrices $\text{diag}(g_1, \dots, g_n) \in G_n$, and

$$(2.20) \quad \pi \cdot (v_1 \otimes \cdots \otimes v_n \otimes u) := v_{\pi^{-1}1} \otimes \cdots \otimes v_{\pi^{-1}n} \otimes (\pi.u),$$

for all $\pi \in \mathfrak{S}_n \subseteq G_n$. If V is an irreducible G -module and U is an irreducible \mathfrak{S}_n -module, then $V \wr U$ is an irreducible G_n -module, but not all of the irreducible G_n -modules arise in this way.

For any composition $\alpha = (\alpha_1, \dots, \alpha_r) \models n$ with r parts, the parabolic subgroup of block diagonal matrices in G_n with block sizes $\alpha_1, \dots, \alpha_r$ gives an inclusion

$$(2.21) \quad G_\alpha := G_{\alpha_1} \times \cdots \times G_{\alpha_r} \subseteq G_n.$$

If W_i is a G_{α_i} -module for $1 \leq i \leq r$, the tensor product $W_1 \otimes \cdots \otimes W_r$ is a G_α -module and the induction $\text{Ind}_{G_\alpha}^{G_n}(W_1 \otimes \cdots \otimes W_r)$ is a G_n -module.

We index the irreducible representations of the cyclic group $G = \mathbb{Z}_r = \langle \zeta \rangle$ in the following slightly nonstandard way. For $1 \leq i \leq r$, let $\rho_i : G \rightarrow GL_1(\mathbb{C}) = \mathbb{C}^\times$ be the homomorphism

$$(2.22) \quad \rho_i : \zeta \mapsto \zeta^{-i}.$$

and let V_i be the vector space \mathbb{C} with G -module structure given by ρ_i . In particular, we have that V_r is the trivial representation of G and V_1, V_2, \dots, V_{r-1} are the nontrivial irreducible representations of G .

The irreducible modules for G_n are indexed by r -partitions of n . If $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)}) \vdash_r n$ is an r -partition of n , let $\alpha = (\alpha_1, \dots, \alpha_r) \models n$ be the composition whose parts are $\alpha_i := |\lambda^{(i)}|$. Define \mathbf{S}^λ to be the G_n -module given by

$$(2.23) \quad \mathbf{S}^\lambda := \text{Ind}_{G_\alpha}^{G_n} ((V_1 \wr S^{\lambda^{(1)}}) \otimes \dots \otimes (V_r \wr S^{\lambda^{(r)}})).$$

Specht proved that the set $\{\mathbf{S}^\lambda : \lambda \vdash_r n\}$ forms a complete set of nonisomorphic irreducible representations of G_n .

Example 2.2. For any $1 \leq i \leq r$, both of the functions

$$(2.24) \quad \begin{cases} \chi^i : g \mapsto (\chi(g))^i \\ \text{sign} \cdot \chi^i : g \mapsto \text{sign}(g) \cdot (\chi(g))^i \end{cases}$$

on G_n are linear characters. We leave it for the reader to check that under the above classification we have

$$\begin{aligned} \chi^1 &\leftrightarrow ((n), \emptyset, \dots, \emptyset), & \text{sign} \cdot \chi^1 &\leftrightarrow ((1^n), \emptyset, \dots, \emptyset), \\ \chi^2 &\leftrightarrow (\emptyset, (n), \dots, \emptyset), & \text{sign} \cdot \chi^2 &\leftrightarrow (\emptyset, (1^n), \dots, \emptyset), \\ &\vdots & & \vdots \\ \chi^r &\leftrightarrow (\emptyset, \emptyset, \dots, (n)), & \text{sign} \cdot \chi^r &\leftrightarrow (\emptyset, \emptyset, \dots, (1^n)). \end{aligned}$$

Since χ^r is the trivial character of G_n , the trivial representation therefore corresponds to the r -partition $(\emptyset, \dots, \emptyset, (n))$.

Let V be a finite-dimensional G_n -module. There exist unique integers m_λ such that

$$V \cong \bigoplus_{\lambda \vdash_r n} (\mathbf{S}^\lambda)^{m_\lambda}.$$

The *Frobenius character* $\text{Frob}(V) \in \Lambda^r(\mathbf{x})$ of V is given by

$$(2.25) \quad \text{Frob}(V) := \sum_{\lambda \vdash_r n} m_\lambda \mathbf{s}_\lambda(\mathbf{x}).$$

In particular, the multiplicity m_λ of \mathbf{S}^λ in V is $\langle \text{Frob}(V), \mathbf{s}_{\lambda^*}(\mathbf{x}) \rangle$.

More generally, if $V = \bigoplus_{d \geq 0} V_d$ is a graded G_n -module with each V_d finite-dimensional, the *graded Frobenius character* $\text{grFrob}(V; q) \in \Lambda^r(\mathbf{x})[[q]]$ of V is

$$(2.26) \quad \text{grFrob}(V; q) := \sum_{d \geq 0} \text{Frob}(V_d) \cdot q^d.$$

Also recall that the *Hilbert series* $\text{Hilb}(V; q)$ of V is

$$(2.27) \quad \text{Hilb}(V; q) := \sum_{d \geq 0} \dim(V_d) \cdot q^d.$$

The Frobenius character is compatible with induction product in the following way. Let V be an G_n -module and let W be a G_m module. The tensor product $V \otimes W$ is a $G_{(n,m)}$ -module, so that $\text{Ind}_{G_{(n,m)}}^{G_{n+m}}(V \otimes W)$ is a G_{n+m} -module. We have

$$(2.28) \quad \text{Frob}(\text{Ind}_{G_{(n,m)}}^{G_{n+m}}(V \otimes W)) = \text{Frob}(V) \cdot \text{Frob}(W),$$

where the multiplication on the right-hand side takes place within $\Lambda^r(\mathbf{x})$.

2.5. Gröbner theory. A total order $<$ on the monomials in $\mathbb{C}[\mathbf{x}_n]$ is called a *monomial order* if

- $1 \leq m$ for every monomial $m \in \mathbb{C}[\mathbf{x}_n]$, and
- $m \leq m'$ implies $m \cdot m'' \leq m' \cdot m''$ for all monomials $m, m', m'' \in \mathbb{C}[\mathbf{x}_n]$.

In this paper we will only use the *lexicographic* monomial order defined by $x_1^{a_1} \cdots x_n^{a_n} < x_1^{b_1} \cdots x_n^{b_n}$ if there exists $1 \leq i \leq n$ such that $a_1 = b_1, \dots, a_{i-1} = b_{i-1}$, and $a_i < b_i$.

If $f \in \mathbb{C}[\mathbf{x}_n]$ is a nonzero polynomial and $<$ is a monomial order, let $\text{in}_<(f)$ be the leading term of f with respect to the order $<$. If $I \subseteq \mathbb{C}[\mathbf{x}_n]$ is an ideal, the corresponding *initial ideal* $\text{in}_<(I) \subseteq \mathbb{C}[\mathbf{x}_n]$ is the monomial ideal in $\mathbb{C}[\mathbf{x}_n]$ generated by the leading terms of every nonzero polynomial in I :

$$(2.29) \quad \text{in}_<(I) := \langle \text{in}_<(f) : f \in I - \{0\} \rangle.$$

The collection of monomials $m \in \mathbb{C}[\mathbf{x}_n]$ which are not contained in $\text{in}_<(I)$, namely

$$(2.30) \quad \{\text{monomials } m \in \mathbb{C}[\mathbf{x}_n] : \text{in}_<(f) \nmid m \text{ for all } f \in I - \{0\}\}$$

descends to a vector space basis for the quotient $\mathbb{C}[\mathbf{x}_n]/I$. This is called the *standard monomial basis*.

A finite subset $B = \{g_1, \dots, g_m\} \subseteq I$ of nonzero polynomials in I is called a *Gröbner basis* of I if $\text{in}_<(I) = \langle \text{in}_<(g_1), \dots, \text{in}_<(g_m) \rangle$. A Gröbner basis B is called *reduced* if

- the leading coefficient of g_i is 1 for all $1 \leq i \leq m$, and
- for $i \neq j$, the monomial $\text{in}_<(g_i)$ does not divide any of the terms appearing in g_j .

After fixing a monomial order, every ideal $I \subseteq \mathbb{C}[\mathbf{x}_n]$ has a unique reduced Gröbner basis.

3. POLYNOMIAL IDENTITIES

In this section we prove a family of polynomial and symmetric function identities which will be useful in our analysis of the rings $R_{n,k}$ and $S_{n,k}$. The first of these identities is the G_n -analogue of [14, Lem. 3.1].

Lemma 3.1. *Let $k \leq n$, let $\alpha_1, \dots, \alpha_k \in \mathbb{C}$ be distinct complex numbers, and let $\beta_1, \dots, \beta_n \in \mathbb{C}$ be complex numbers with the property that $\{\alpha_1, \dots, \alpha_k\} \subseteq \{\beta_1^r, \dots, \beta_n^r\}$. For any $n - k + 1 \leq s \leq n$ we have*

$$(3.1) \quad \sum_{j=0}^s (-1)^j e_{s-j}(\beta_1^r, \dots, \beta_n^r) h_j(\alpha_1, \dots, \alpha_k) = 0.$$

Proof. The left-hand side is the coefficient of t^s in the power series

$$(3.2) \quad \frac{\prod_{i=1}^n (1 + t\beta_i^r)}{\prod_{i=1}^k (1 + t\alpha_i)}.$$

By assumption, every term in the denominator cancels with a distinct term in the numerator, so that this expression is a polynomial in t of degree $n - k$. Since $s > n - k$, the coefficient of t^s in this polynomial is 0. \square

In practice, our applications of Lemma 3.1 will always involve one of the two situations $\{\beta_1^r, \dots, \beta_n^r\} = \{\alpha_1, \dots, \alpha_k\}$ or $\{\beta_1^r, \dots, \beta_n^r\} = \{\alpha_1, \dots, \alpha_k, 0\}$.

Let $\gamma = (\gamma_1, \dots, \gamma_n) \models n$ be a composition with n parts. The *Demazure character* $\kappa_\gamma(\mathbf{x}_n) \in \mathbb{C}[\mathbf{x}_n]$ is defined recursively as follows. If $\gamma_1 \geq \dots \geq \gamma_n$, we let $\kappa_\gamma(\mathbf{x}_n)$ be the monomial

$$(3.3) \quad \kappa_\gamma(\mathbf{x}_n) = x_1^{\gamma_1} \cdots x_n^{\gamma_n}.$$

In general, if $\gamma_i < \gamma_{i+1}$, we let

$$(3.4) \quad \kappa_\gamma(\mathbf{x}_n) = \frac{x_i(\kappa_{\gamma'}(\mathbf{x}_n)) - x_{i+1}(s_i \cdot \kappa_{\gamma'}(\mathbf{x}_n))}{x_i - x_{i+1}},$$

where $\gamma' = (\gamma_1, \dots, \gamma_{i+1}, \gamma_i, \dots, \gamma_n)$ is the composition obtained by interchanging the i^{th} and $(i+1)^{\text{st}}$ parts of γ and $s_i \cdot \kappa_{\gamma'}(\mathbf{x}_n)$ is the polynomial $\kappa_{\gamma'}(\mathbf{x}_n)$ with x_i and x_{i+1} interchanged. It can be shown that this recursion gives a well defined collection of polynomials $\{\kappa_\gamma(\mathbf{x}_n)\}$ indexed by compositions γ with n parts. This set forms a basis for the polynomial ring $\mathbb{C}[\mathbf{x}_n]$.

Demazure characters played a key role in [14]; they will be equally important here. In order to state the G_n -analogs of the lemmata from [14] that we will need, we must introduce some notation.

Definition 3.2. Let $S = \{s_1 < s_2 < \dots < s_m\} \subseteq [n]$. The *skip monomial* $\mathbf{x}(S)$ in $\mathbb{C}[\mathbf{x}_n]$ is

$$\mathbf{x}(S) := x_{s_1}^{s_1} x_{s_2}^{s_2-1} \dots x_{s_m}^{s_m-m+1}.$$

The *skip composition* $\gamma(S) = (\gamma_1, \dots, \gamma_n)$ is the length n composition defined by

$$\gamma_i = \begin{cases} 0 & i \notin S \\ s_j - j + 1 & i = s_j \in S. \end{cases}$$

We also let $\overline{\gamma(S)} := (\gamma_n, \dots, \gamma_1)$ be the reverse of the skip composition $\gamma(S)$.

For example, if $n = 8$ and $S = \{2, 3, 5, 8\}$, then $\gamma(S) = (0, 2, 2, 0, 3, 0, 0, 5)$ and $\mathbf{x}(S) = x_2^2 x_3^2 x_5^3 x_8^5$. In general, we have that $\gamma(S)$ is the exponent vector of $\mathbf{x}(S)$. We will be interested in the r^{th} powers $\mathbf{x}(S)^r$ of skip monomials in this paper.

Skip monomials are related to Demazure characters as follows. For any polynomial $f(\mathbf{x}_n) = f(x_1, \dots, x_n) \in \mathbb{C}[\mathbf{x}_n]$, let $f(\mathbf{x}_n^r) = f(x_1^r, \dots, x_n^r)$ and $\overline{f(\mathbf{x}_n^r)} = f(x_n^r, \dots, x_1^r)$. The following result is immediate from [14, Lem. 3.5] after the change of variables $(x_1, \dots, x_n) \mapsto (x_1^r, \dots, x_n^r)$.

Lemma 3.3. Let $n \geq k$ and let $S \subseteq [n]$ satisfy $|S| = n - k + 1$. Let $<$ be lexicographic order. We have

$$(3.5) \quad \text{in}_{<}(\overline{\kappa_{\gamma(S)}(\mathbf{x}_n^r)}) = \mathbf{x}(S)^r.$$

Moreover, for any $1 \leq i \leq n$ we have

$$(3.6) \quad x_i^{r \cdot (\max(S) - n + k + 1)} \nmid m$$

for any monomial m appearing in $\overline{\kappa_{\gamma(S)}(\mathbf{x}_n^r)}$. Finally, if $T \subseteq [n]$ satisfies $|T| = n - k + 1$ and $T \neq S$, then $\mathbf{x}(S)^r \nmid m$ for any monomial m appearing in $\overline{\kappa_{\gamma(T)}(\mathbf{x}_n^r)}$.

We also record the fact, which follows immediately from [14], that the polynomials $\kappa_{\gamma(S)^*}(\mathbf{x}_n^{r,*})$ appearing in Lemma 3.3 are contained in the ideals $I_{n,k}$ and $J_{n,k}$. The following result follows from [14, Eqn. 3.4] after the change of variables $(x_1, \dots, x_n) \mapsto (x_1^r, \dots, x_n^r)$.

Lemma 3.4. Let $n \geq k$ and let $S \subseteq [n]$ satisfy $|S| = n - k + 1$. The polynomial $\overline{\kappa_{\gamma(S)}(\mathbf{x}_n^r)}$ is contained in the ideal

$$(3.7) \quad \langle e_n(\mathbf{x}_n^r), e_{n-1}(\mathbf{x}_n^r), \dots, e_{n-k+1}(\mathbf{x}_n^r) \rangle \subseteq \mathbb{C}[\mathbf{x}_n].$$

In particular, we have $\overline{\kappa_{\gamma(S)}(\mathbf{x}_n^r)} \in I_{n,k}$ and $\overline{\kappa_{\gamma(S)}(\mathbf{x}_n^r)} \in J_{n,k}$.

We define two formal power series in the infinite variable set $\mathbf{x} = (\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(r)})$ using the coinvariant and comaj statistics on r -colored ordered multiset partitions. If μ is an r -colored ordered multiset partition, let \mathbf{x}^μ be the monomial in the variable set \mathbf{x} where the exponent of $x_j^{(i)}$ is the number of occurrences of j^{i-1} in μ .

Definition 3.5. Let $r \geq 1$ and let $k \leq n$ be positive integers. Define two formal power series in the variable set $\mathbf{x} = (\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(r)})$ by

$$(3.8) \quad M_{n,k}(\mathbf{x}; q) := \sum_{\mu} q^{\text{maj}(\mu)} \mathbf{x}^{\mu},$$

$$(3.9) \quad I_{n,k}(\mathbf{x}; q) := \sum_{\mu} q^{\text{coinv}(\mu)} \mathbf{x}^{\mu},$$

where the sum is over all r -colored ordered multiset partitions μ of size n with k blocks.

The next result establishes that the formal power series $M_{n,k}(\mathbf{x}; q)$, $I_{n,k}(\mathbf{x}; q)$ in Definition 3.5 both contained in the ring $\Lambda^r(\mathbf{x})$ and are related to each other by q -reversal.

Lemma 3.6. *Both of the formal power series $M_{n,k}(\mathbf{x}; q)$ and $I_{n,k}(\mathbf{x}; q)$ lie in the ring $\Lambda^r(\mathbf{x})$. Moreover, we have $M_{n,k}(\mathbf{x}; q) = \text{rev}_q(I_{n,k}(\mathbf{x}; q))$.*

Proof. The truth of this statement for $r = 1$ (when $\Lambda^r(\mathbf{x})$ is the usual ring of symmetric functions) follows from the work of Wilson [24]. To deduce this statement for general $r \geq 1$, consider a new countably infinite set of variables

$$(3.10) \quad \mathbf{z} = \{z_{i,j} : j \in \mathbb{Z}_{>0}, 1 \leq i \leq r\}.$$

The association $z_{i,j} \leftrightarrow x_j^{(i)}$ gives a bijection with our collection of variables $\mathbf{x} = (\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(r)})$. The idea is to reinterpret $M_{n,k}(\mathbf{x}; q)$ and $I_{n,k}(\mathbf{x}; q)$ in terms of the new variable set \mathbf{z} , and then apply the equality and symmetry known in the case $r = 1$.

To achieve the program of the preceding paragraph, we introduce the following notation. Let $M_{n,k}^1(\mathbf{z}; q^r)$ be the formal power series

$$(3.11) \quad M_{n,k}^1(\mathbf{z}; q^r) := \sum_{\mu} q^{r \cdot \text{maj}(\mu)} \mathbf{z}^{\mu},$$

where the sum is over all ordered multiset partitions μ of size n with k blocks on the countably infinite alphabet

$$1^{r-1} < 2^{r-1} < \dots < 1^{r-2} < 2^{r-2} < \dots < 1^0 < 2^0 < \dots$$

and we compute $\text{maj}(\mu)$ as in the $r = 1$ case (i.e., ignoring contributions to maj coming from colors, and not multiplying descents by r).

Similarly, let $I_{n,k}^1(\mathbf{z}; q^r)$ be the formal power series

$$(3.12) \quad I_{n,k}^1(\mathbf{z}; q) := \sum_{\mu} q^{r \cdot \text{coinv}(\mu)} \mathbf{z}^{\mu},$$

where the sum is over all ordered multiset partitions μ of size n with k blocks on the countably infinite alphabet

$$1^{r-1} \prec \dots \prec 1^0 \prec 2^{r-1} \prec \dots \prec 2^0 \prec \dots$$

and we define $\text{coinv}(\mu)$ as in the $r = 1$ case (i.e., ignoring the contribution to coinv coming from colors, and not multiplying the number of coinversion pairs by r).

It follows from the definition of $M_{n,k}(\mathbf{x}; q)$ that

$$(3.13) \quad M_{n,k}(\mathbf{x}; q) = M_{n,k}^1(\mathbf{z}; q^r) \big|_{z_{i,j} = q^{i-1} \cdot x_j^{(i)}}.$$

This expression for $M_{n,k}(\mathbf{x}; q)$, together with the fact that $M_{n,k}^1(\mathbf{z}; q^r)$ is symmetric in the \mathbf{z} variables, proves that $M_{n,k}(\mathbf{x}; q) \in \Lambda^r(\mathbf{x})$. Similarly, we have

$$(3.14) \quad I_{n,k}(\mathbf{x}; q) = I_{n,k}^1(\mathbf{z}; q^r) \big|_{z_{i,j} = q^{r-i} \cdot x_j^{(i)}},$$

so that $I_{n,k}(\mathbf{x}; q) \in \Lambda^r(\mathbf{x})$.

Applying the lemma in the case $r = 1$, we have

$$(3.15) \quad M_{n,k}(\mathbf{x}; q) = M_{n,k}^1(z_{1,r}, z_{2,r}, \dots, z_{1,r-1}, z_{2,r-1}, \dots, z_{1,1}, z_{2,1}, \dots; q^r) \Big|_{z_{i,j}=q^{i-1} \cdot x_j^{(i)}}$$

$$(3.16) \quad = M_{n,k}^1(z_{1,r}, \dots, z_{1,1}, z_{2,r}, \dots, z_{2,1}, \dots; q^r) \Big|_{z_{i,j}=q^{i-1} \cdot x_j^{(i)}}$$

$$(3.17) \quad = \text{rev}_q \left[I_{n,k}^1(z_{1,r}, \dots, z_{1,1}, z_{2,r}, \dots, z_{2,1}, \dots; q^r) \right] \Big|_{z_{i,j}=q^{i-1} \cdot x_j^{(i)}}$$

$$(3.18) \quad = \text{rev}_q \left[I_{n,k}^1(z_{1,r}, \dots, z_{1,1}, z_{2,r}, \dots, z_{2,1}, \dots; q^r) \Big|_{z_{i,j}=q^{r-i} \cdot x_j^{(i)}} \right]$$

$$(3.19) \quad = \text{rev}_q(I_{n,k}(\mathbf{x}; q)).$$

The first equality is Equation 3.13, the second equality uses the fact that $M_{n,k}^1(\mathbf{z}; q)$ is symmetric in the \mathbf{z} variables, the third equality uses the fact that $M_{n,k}^1(\mathbf{z}; q) = \text{rev}_q(I_{n,k}^1(\mathbf{z}; q))$, the fourth equality interchanges evaluation and q -reversal, and the final equality is Equation 3.14. \square

The power series in Lemma 3.6 will be (up to minor transformations) the graded Frobenius character of the ring $S_{n,k}$. We give this character-to-be a name.

Definition 3.7. Let $r \geq 1$ and let $k \leq n$ be positive integers. Let $D_{n,k}(\mathbf{x}; q) \in \Lambda^r(\mathbf{x})$ be the common ring element

$$(3.20) \quad D_{n,k}(\mathbf{x}; q) := (\text{rev}_q \circ \omega) M_{n,k}(\mathbf{x}; q) = \omega I_{n,k}(\mathbf{x}; q).$$

As a Frobenius character, the ring element $D_{n,k}(\mathbf{x}; q) \in \Lambda^r(\mathbf{x})$ must expand positively in the Schur basis $\{s_\lambda(\mathbf{x}) : \lambda \vdash_r n\}$. The maj formulation of $D_{n,k}(\mathbf{x}; q)$ is well suited to proving this fact directly, as well as giving the Schur expansion of $D_{n,k}(\mathbf{x}; q)$. The following proposition is a colored version of a result of Wilson [24, Thm. 5.0.1].

Proposition 3.8. Let $r \geq 1$ and let $k \leq n$ be positive integers. We have the Schur expansion

$$(3.21) \quad D_{n,k}(\mathbf{x}; q) = \text{rev}_q \left[\sum_{T \in \text{SYT}^r(n)} q^{\text{maj}(T) + r \binom{n-k}{2} - r(n-k)\text{des}(T)} \begin{bmatrix} \text{des}(T) \\ n-k \end{bmatrix}_{q^r} s_{\text{shape}(T)'}(\mathbf{x}) \right].$$

Proof. Consider the collection \mathcal{W}_n of all length n words $w = w_1 \dots w_n$ in the alphabet of r -colored positive integers. For any word $w \in \mathcal{W}_n$, the (colored version of the) *RSK correspondence* gives a pair of r -tableaux (U, T) of the same shape, with U semistandard and T standard. For example, if $r = 3$ and $w = 2^0 1^1 4^1 2^2 1^0 2^0 2^1 1^2 \in \mathcal{W}_8$ then $w \mapsto (U, T)$ where

$$U = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 4 & \\ \hline \end{array}, \quad \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} \quad T = \begin{array}{|c|c|} \hline 1 & 6 \\ \hline 5 & \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 7 & \\ \hline \end{array}, \quad \begin{array}{|c|} \hline 4 \\ \hline 8 \\ \hline \end{array}.$$

The RSK map gives a bijection

$$(3.22) \quad \mathcal{W}_n \xrightarrow{\sim} \left\{ (U, T) : \begin{array}{l} U \text{ a semistandard } r\text{-tableau with } n \text{ boxes,} \\ T \text{ a standard } r\text{-tableau with } n \text{ boxes,} \\ \text{shape}(U) = \text{shape}(T) \end{array} \right\}.$$

If $w \mapsto (U, T)$, then $\text{Des}(w) = \text{Des}(T)$ so that $\text{maj}(w) = \text{maj}(T)$.

For any word $w \in \mathcal{W}_n$, we can generate a collection of $\binom{\text{des}(w)}{n-k}$ r -colored ordered multiset partitions μ as follows. Among the $\text{des}(w)$ descents of w , choose $n-k$ of them to star, yielding a pair (w, S) where $S \subseteq \text{Des}(w)$ satisfies $|S| = n-k$. We may identify (w, S) with an r -colored ordered multiset partition μ .

The above paragraph implies that

$$(3.23) \quad M_{n,k}(\mathbf{x}; q) = \sum_{w \in \mathcal{W}_n^r} q^{\text{maj}(w) + r \binom{n-k}{2} - r(n-k)\text{des}(w)} \begin{bmatrix} \text{des}(w) \\ n-k \end{bmatrix}_{q^r} \mathbf{x}^w,$$

where the factor $q^{r\binom{n-k}{2}-r(n-k)\text{des}(w)} \left[\begin{smallmatrix} \text{des}(w) \\ n-k \end{smallmatrix} \right]_{q^r}$ is generated by the ways in which $n-k$ stars can be placed among the $\text{des}(w)$ descents of w .

Applying RSK to the right-hand side of Equation 3.23, we deduce that

$$(3.24) \quad M_{n,k}(\mathbf{x}; q) = \sum_{T \in \text{SYT}^r(n)} q^{\text{maj}(T) - r\binom{n-k}{2} + r(n-k)\text{des}(T)} \left[\begin{smallmatrix} \text{des}(T) \\ n-k \end{smallmatrix} \right]_{q^r} s_{\text{shape}(T)}(\mathbf{x}).$$

Since $D_{n,k}(\mathbf{x}; q) = (\text{rev}_q \circ \omega) M_{n,k}(\mathbf{x}; q)$, we are done. \square

Our basic tool for proving that $D_{n,k}(\mathbf{x}; q) = \text{grFrob}(S_{n,k}; q)$ will be the following lemma, which is a colored version of [14, Lem. 3.6].

Lemma 3.9. *Let $\mathbf{F}(\mathbf{x}), \mathbf{G}(\mathbf{x}) \in \Lambda^r(\mathbf{x})$ have equal constant terms. Then $\mathbf{F}(\mathbf{x}) = \mathbf{G}(\mathbf{x})$ if and only if $e_j(\mathbf{x}^{(i*)})^\perp \mathbf{F}(\mathbf{x}) = e_j(\mathbf{x}^{(i*)})^\perp \mathbf{G}(\mathbf{x})$ for all $j \geq 1$ and $1 \leq i \leq r$.*

Proof. The forward direction is obvious. For the reverse direction, let λ be any r -partition, let $j \geq 1$, and let $1 \leq i \leq r$. We have

$$(3.25) \quad \langle \mathbf{F}(\mathbf{x}), e_j(\mathbf{x}^{(i*)}) e_\lambda(\mathbf{x}) \rangle = \langle e_j(\mathbf{x}^{(i*)})^\perp \mathbf{F}(\mathbf{x}), e_\lambda(\mathbf{x}) \rangle$$

$$(3.26) \quad = \langle e_j(\mathbf{x}^{(i*)})^\perp \mathbf{G}(\mathbf{x}), e_\lambda(\mathbf{x}) \rangle$$

$$(3.27) \quad = \langle \mathbf{G}(\mathbf{x}), e_j(\mathbf{x}^{(i*)}) e_\lambda(\mathbf{x}) \rangle.$$

Since $\langle \mathbf{F}(\mathbf{x}), e_\emptyset(\mathbf{x}) \rangle = \langle \mathbf{G}(\mathbf{x}), e_\emptyset(\mathbf{x}) \rangle$ by assumption (where $\emptyset = (\emptyset, \dots, \emptyset)$ is the empty r -partition), this chain of equalities implies that $\langle \mathbf{F}(\mathbf{x}), e_\lambda(\mathbf{x}) \rangle = \langle \mathbf{G}(\mathbf{x}), e_\lambda(\mathbf{x}) \rangle$ for any r -partition λ . We conclude that $\mathbf{F}(\mathbf{x}) = \mathbf{G}(\mathbf{x})$. \square

We will show that $D_{n,k}(\mathbf{x}; q)$ and $\text{grFrob}(S_{n,k}; q)$ satisfy the conditions of Lemma 3.9 by showing that their images under $e_j(\mathbf{x}^{(i*)})^\perp$ satisfy the same recursion. The coinvariant formulation of $D_{n,k}(\mathbf{x}; q)$ is best suited to calculating $e_j(\mathbf{x}^{(i*)})^\perp$. The following lemma is a colored version of [14, Lem. 3.7].

Lemma 3.10. *Let $r \geq 1$ and let $k \leq n$ be positive integers. Let $1 \leq i \leq r$ and let $j \geq 1$. We have*

$$(3.28) \quad e_j(\mathbf{x}^{(i*)})^\perp D_{n,k}(\mathbf{x}; q) = q^{j \cdot (r-i) + r \cdot \binom{j}{2}} \left[\begin{smallmatrix} k \\ j \end{smallmatrix} \right]_{q^r} \sum_{m=\max(1, k-j)}^{\min(k, n-j)} q^{r \cdot (k-m) \cdot (n-j-m)} \left[\begin{smallmatrix} j \\ k-m \end{smallmatrix} \right]_{q^r} D_{n-j, m}(\mathbf{x}; q).$$

Proof. Applying ω to both sides of the purported identity, it suffices to prove

$$(3.29) \quad h_j(\mathbf{x}^{(i*)})^\perp I_{n,k}(\mathbf{x}; q) = q^{j \cdot (r-i) + r \cdot \binom{j}{2}} \left[\begin{smallmatrix} k \\ j \end{smallmatrix} \right]_{q^r} \sum_{m=\max(1, k-j)}^{\min(k, n-j)} q^{r \cdot (k-m) \cdot (n-j-m)} \left[\begin{smallmatrix} j \\ k-m \end{smallmatrix} \right]_{q^r} I_{n-j, m}(\mathbf{x}; q).$$

Since the bases $\{h_\lambda(\mathbf{x})\}$ and $\{m_{\lambda^*}(\mathbf{x})\}$ are dual bases for $\Lambda^r(\mathbf{x})$ under the Hall inner product, for any $\mathbf{F}(\mathbf{x}) \in \Lambda^r(\mathbf{x})$ and any r -composition β , we have

$$(3.30) \quad \langle \mathbf{F}(\mathbf{x}), h_{\beta^*}(\mathbf{x}) \rangle = \text{coefficient of } \mathbf{x}^\beta \text{ in } \mathbf{F}(\mathbf{x}).$$

Equation 3.30 is our tool for proving Equation 3.29.

Let $\beta = (\beta^{(1)}, \dots, \beta^{(r)})$ be an r -composition and consider the inner product

$$(3.31) \quad \langle h_j(\mathbf{x}^{(i*)})^\perp I_{n,k}(\mathbf{x}; q), h_{\beta^*}(\mathbf{x}) \rangle = \langle I_{n,k}(\mathbf{x}; q), h_j(\mathbf{x}^{(i*)}) h_{\beta^*}(\mathbf{x}) \rangle.$$

We may write $h_j(\mathbf{x}^{(i*)}) h_{\beta^*}(\mathbf{x}) = h_{\widehat{\beta}^*}(\mathbf{x})$, where

- $\widehat{\beta} = (\beta^{(1)}, \dots, \widehat{\beta}^{(i)}, \dots, \beta^{(r)})$ is an r -composition which agrees with β in every component except for i , and

- $\widehat{\beta}^{(i)} = (\beta_1^{(i)}, \beta_2^{(i)}, \dots, 0, \dots, 0, j)$, where the composition $\widehat{\beta}^{(i)}$ has N parts for some positive integer N larger than the number of parts in any of $\beta^{(1)}, \dots, \beta^{(r)}$.

By Equation 3.30, we can interpret $\langle \mathbf{I}_{n,k}(\mathbf{x}), h_j(\mathbf{x}^{(i)}) \mathbf{h}_{\beta^*}(\mathbf{x}) \rangle = \langle \mathbf{I}_{n,k}(\mathbf{x}), \mathbf{h}_{\widehat{\beta}^*}(\mathbf{x}) \rangle$ combinatorially.

For any r -composition $\alpha = (\alpha^{(1)}, \dots, \alpha^{(r)})$, let $\mathcal{OP}_{\alpha,k}$ be the collection of r -colored ordered multiset partitions with k blocks which contain $\alpha_j^{(i)}$ copies of the letter j^{i-1} . Equation 3.30 implies

$$(3.32) \quad \langle \mathbf{I}_{n,k}(\mathbf{x}), \mathbf{h}_{\widehat{\beta}^*}(\mathbf{x}) \rangle = \sum_{\mu \in \mathcal{OP}_{\widehat{\beta},k}} q^{\text{coinv}(\mu)}.$$

Let us analyze the right-hand side of Equation 3.32. A typical element $\mu \in \mathcal{OP}_{\widehat{\beta},k}$ contains j copies of the *big letter* N^{i-1} , together with various other *small letters*. Recall that the statistic coinv is defined using the order \prec , which prioritizes letter value over color. Our choice of N guarantees that every small letter is $\prec N^{i-1}$. We have a map

$$(3.33) \quad \varphi : \mathcal{OP}_{\widehat{\beta},k} \rightarrow \bigcup_{m=\max(1,k-j)}^{\min(k,n-j)} \mathcal{OP}_{\beta,m},$$

where $\varphi(\mu)$ is the r -colored ordered multiset partition obtained by erasing all j of the big letters N^{i-1} in μ (together with any singleton blocks $\{N^{i-1}\}$). Let us analyze the effect of φ on coinv .

Fix m in the range $\max(1, k-j) \leq m \leq \min(k, n-j)$ and let $\mu \in \mathcal{OP}_{\beta,m}$. Then any $\mu' \in \varphi^{-1}(\mu)$ is obtained by adding j copies of the big letter N^{i-1} to μ , precisely $k-m$ of which must be added in singleton blocks. We calculate $\sum_{\mu' \in \varphi^{-1}(\mu)} q^{\text{coinv}(\mu')}$ in terms of $\text{coinv}(\mu)$ as follows.

Following the notation of the proof of [14, Lem. 3.7], let us call a big letter N^{i-1} *minb* if it is \prec -minimal in its block and *nminb* if it is not \prec -minimal in its block. Similarly, let us call a small letter *mins* or *nmins* depending on whether it is minimal in its block. The contributions to $\sum_{\mu' \in \varphi^{-1}(\mu)} q^{\text{coinv}(\mu')}$ coming from big letters are as follows.

- The j big letters N^{i-1} give a complementary color contribution of $j \cdot (r-i)$ to coinv .
- Each of the *minb* letters forms a coinversion pair with every *nmins* letter. Since there are $k-m$ *minb* letters and $n-j-m$ *nmins* letters, this contributes $r(k-m)(n-j-m)$ to coinv .
- Each of the *minb* letters forms a coinversion pair with every *nminb* letter (for a total of $(k-m)(j-k+m)$ coinversion pairs) as well each *minb* letter to its left (for a total of $\binom{k-m}{2}$ coinversion pairs). This contributes $r \cdot [(k-m)(j-k+m) + \binom{k-m}{2}]$ to coinv .
- Each *minb* letter forms a coinversion pair with each *mins* letter to its left. If we sum over the $\binom{k}{k-m}$ ways of interleaving the singleton blocks $\{N^{i-1}\}$ within the blocks of μ , this gives rise to a factor of $\left[\binom{k}{k-m} \right]_{q^r}$.
- Each *nminb* letter forms a coinversion pair with each *mins* letter to its left. If we consider the $\binom{m}{j-k+m}$ ways to augment the m blocks of μ with a *nminb* letter, this gives rise to a factor of $q^{\binom{j-k+m}{2}} \left[\binom{m}{j-k+m} \right]_{q^r}$.

Applying the identity

$$(3.34) \quad r \cdot \left[(k-m)(j-k-m) + \binom{k-m}{2} + \binom{j-k+m}{2} \right] = r \cdot \binom{j}{2},$$

we see that

$$(3.35) \quad \sum_{\mu' \in \varphi^{-1}(\mu)} q^{\text{coinv}(\mu')} = q^{j \cdot (r-i) + r \cdot \binom{j}{2} + r \cdot (k-m)(n-j-m)} \begin{bmatrix} k \\ k-m \end{bmatrix}_{q^r} \begin{bmatrix} m \\ j-k+m \end{bmatrix}_{q^r} q^{\text{coinv}(\mu)}$$

$$(3.36) \quad = q^{j \cdot (r-i) + r \cdot \binom{j}{2} + r \cdot (k-m)(n-j-m)} \begin{bmatrix} k \\ j \end{bmatrix}_{q^r} \begin{bmatrix} j \\ k-m \end{bmatrix}_{q^r} q^{\text{coinv}(\mu)}.$$

If we sum this expression over all $\mu \in \mathcal{OP}_{\beta, m}$, and then sum over m , we get

$$(3.37) \quad q^{j \cdot (r-i) + r \cdot \binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{q^r} \sum_{m=\max(1, k-j)}^{\min(k, n-j)} q^{r \cdot (k-m)(n-j-m)} \begin{bmatrix} j \\ k-m \end{bmatrix}_{q^r} \sum_{\mu \in \mathcal{OP}_{\beta, m}} q^{\text{coinv}(\mu)}.$$

However, thanks to Equation 3.30 and the definition of the \mathbf{I} -functions, the expression (3.37) is also equal to

$$(3.38) \quad \left\langle q^{j \cdot (r-i) + r \cdot \binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{q^r} \sum_{m=\max(1, k-j)}^{\min(k, n-j)} q^{r \cdot (k-m)(n-j-m)} \begin{bmatrix} j \\ k-m \end{bmatrix}_{q^r} \mathbf{I}_{n-j, m}(\mathbf{x}; q), \mathbf{h}_{\beta^*}(\mathbf{x}) \right\rangle.$$

Since both sides of the equation in the statement of the lemma have the same pairing under $\langle \cdot, \cdot \rangle$ with $\mathbf{h}_{\beta^*}(\mathbf{x})$ for any r -composition β , we are done. \square

4. HILBERT SERIES AND STANDARD MONOMIAL BASIS

4.1. The point sets $Y_{n,k}^r$ and $Z_{n,k}^r$. In this section we derive the Hilbert series of $R_{n,k}$ and $S_{n,k}$. We also prove that, as ungraded G_n -modules, we have $R_{n,k} \cong \mathbb{C}[\mathcal{F}_{n,k}]$ and $S_{n,k} \cong \mathbb{C}[\mathcal{OP}_{n,k}]$. To do this, we will use a general method dating back to Garsia and Procesi [9] in the context of the Tanisaki ideal. We recall the method, and then apply it to our situation.

For any finite point set $Y \subset \mathbb{C}^n$, let $\mathbf{I}(Y) \subseteq \mathbb{C}[\mathbf{x}_n]$ be the ideal of polynomials which vanish on Y . That is, we have

$$(4.1) \quad \mathbf{I}(Y) := \{f \in \mathbb{C}[\mathbf{x}_n] : f(\mathbf{y}) = 0 \text{ for all } \mathbf{y} \in Y\}.$$

We can identify the quotient $\mathbb{C}[\mathbf{x}_n]/\mathbf{I}(Y)$ with the \mathbb{C} -vector space of functions $Y \rightarrow \mathbb{C}$. In particular

$$(4.2) \quad \dim(\mathbb{C}[\mathbf{x}_n]/\mathbf{I}(Y)) = |Y|.$$

If $W \subseteq GL_n(\mathbb{C})$ is a finite subgroup and Y is stable under the action of W , we have

$$(4.3) \quad \mathbb{C}[\mathbf{x}_n]/\mathbf{I}(Y) \cong_W \mathbb{C}[Y]$$

as W -modules, where we used the fact that the permutation module Y is self-dual.

The ideal $\mathbf{I}(Y)$ is almost never homogeneous. To get a homogeneous ideal, we proceed as follows. If $f \in \mathbb{C}[\mathbf{x}_n]$ is any nonzero polynomial of degree d , write $f = f_d + f_{d-1} + \cdots + f_0$, where f_i is homogeneous of degree i . Define $\tau(f) := f_d$ and define a homogeneous ideal $\mathbf{T}(Y) \subseteq \mathbb{C}[\mathbf{x}_n]$ by

$$(4.4) \quad \mathbf{T}(Y) := \langle \tau(f) : f \in \mathbf{I}(Y) - \{0\} \rangle.$$

The passage from $\mathbf{I}(Y)$ to $\mathbf{T}(Y)$ does not affect the W -module structure (or vector space dimension) of the quotient:

$$(4.5) \quad \mathbb{C}[\mathbf{x}_n]/\mathbf{T}(Y) \cong_W \mathbb{C}[\mathbf{x}_n]/\mathbf{I}(Y) \cong_W \mathbb{C}[Y].$$

Our strategy, whose $r = 1$ avatar was accomplished in [14], is as follows.

- (1) Find finite point sets $Y_{n,k}, Z_{n,k} \subset \mathbb{C}^n$ which are stable under the action of G_n such that there are equivariant bijections $Y_{n,k} \cong \mathcal{F}_{n,k}$ and $Z_{n,k} \cong \mathcal{OP}_{n,k}$.
- (2) Prove that $I_{n,k} \subseteq \mathbf{T}(Y_{n,k})$ and $J_{n,k} \subseteq \mathbf{T}(Z_{n,k})$ by showing that the generators of the ideals $I_{n,k}, J_{n,k}$ arise as top degree components of polynomials vanishing on $Y_{n,k}, Z_{n,k}$ (respectively).

(3) Use Gröbner theory to prove

$$\dim(R_{n,k}) = \dim(\mathbb{C}[\mathbf{x}_n]/I_{n,k}) \leq |\mathcal{F}_{n,k}| = \dim(\mathbb{C}[\mathbf{x}_n]/\mathbf{T}(Y_{n,k}))$$

and

$$\dim(S_{n,k}) = \dim(\mathbb{C}[\mathbf{x}_n]/J_{n,k}) \leq |\mathcal{OP}_{n,k}| = \dim(\mathbb{C}[\mathbf{x}_n]/\mathbf{T}(Z_{n,k})).$$

Step 2 then implies $I_{n,k} = \mathbf{T}(Y_{n,k})$ and $J_{n,k} = \mathbf{T}(Z_{n,k})$.

To accomplish Step 1 of this program, we introduce the following point sets.

Definition 4.1. Fix k distinct positive real numbers $0 < \alpha_1 < \dots < \alpha_k$. Let $Y_{n,k} \subset \mathbb{C}^n$ be the set of points (y_1, \dots, y_n) such that

- we have $y_i = 0$ or $y_i \in \{\zeta^c \alpha_j : 0 \leq c \leq r-1, 1 \leq j \leq k\}$ for all i , and
- we have $\{\alpha_1, \dots, \alpha_k\} \subseteq \{|y_1|, \dots, |y_n|\}$.

Let $Z_{n,k} \subseteq \mathbb{C}^n$ be the set of points in $Y_{n,k}$ whose coordinates do not vanish:

$$Z_{n,k} := \{(y_1, \dots, y_n) \in Y_{n,k} : y_i \neq 0 \text{ for all } i\}.$$

There is a bijection $\varphi : \mathcal{F}_{n,k} \rightarrow Y_{n,k}$ given as follows. Let $\sigma = (Z \mid B_1 \mid \dots \mid B_k) \in \mathcal{F}_{n,k}$ be an G_n -face of dimension k , whose zero block Z may be empty. The point $\varphi(\sigma) = (y_1, \dots, y_n)$ has coordinates given by

$$(4.6) \quad y_i = \begin{cases} 0 & \text{if } i \in Z, \\ \zeta^c \alpha_j & \text{if } i \in B_j \text{ and } i \text{ has color } c. \end{cases}$$

For example if $r = 3$ then

$$\varphi : (25 \mid 3^0 \mid 1^0 4^2 6^2) \mapsto (\zeta^0 \alpha_2, 0, \zeta^0 \alpha_1, \zeta^2 \alpha_2, 0, \zeta^2 \alpha_2).$$

The set $Y_{n,k}$ is closed under the action of G_n and the map φ commutes with the action of G_n . It follows that $Y_{n,k} \cong \mathcal{F}_{n,k}$ as G_n -sets. Moreover, the bijection φ restricts to show that $Z_{n,k} \cong \mathcal{OP}_{n,k}$ as G_n -sets. This accomplishes Step 1 of our program.

Step 2 of our program is accomplished by appropriate modifications of [14, Sec. 4].

Lemma 4.2. *We have $I_{n,k} \subseteq \mathbf{T}(Y_{n,k})$ and $J_{n,k} \subseteq \mathbf{T}(Z_{n,k})$.*

Proof. We will show that every generator of $I_{n,k}$ (resp. $J_{n,k}$) is the top degree component of some polynomial in $\mathbf{I}(Y_{n,k})$ (resp. $\mathbf{I}(Z_{n,k})$).

Let $1 \leq i \leq n$. It is clear that $x_i(x_i^r - \alpha_1^r) \cdots (x_i^r - \alpha_k^r) \in \mathbf{I}(Y_{n,k})$. Taking the highest component, we have $x_i^{kr+1} \in \mathbf{T}(Y_{n,k})$. Similarly, the polynomial $(x_i^r - \alpha_1^r) \cdots (x_i^r - \alpha_k^r)$ vanishes on $Z_{n,k}$, so that $x_i^{kr} \in \mathbf{T}(Z_{n,k})$. Lemma 3.1 applies to show $e_{n-k+1}(\mathbf{x}_n^r), \dots, e_n(\mathbf{x}_n^r) \in \mathbf{T}(Y_{n,k})$ and $e_{n-k+1}(\mathbf{x}_n^r), \dots, e_n(\mathbf{x}_n^r) \in \mathbf{T}(Z_{n,k})$. \square

4.2. Skip monomials and initial terms. Step 3 of our program takes more work. We begin by isolating certain monomials in the initial ideals of $I_{n,k}$ and $J_{n,k}$.

Lemma 4.3. *Let $<$ be the lexicographic order on monomials in $\mathbb{C}[\mathbf{x}_n]$.*

- For any $1 \leq i \leq n$ we have $x_i^{kr+1} \in \text{in}_{<}(I_{n,k})$ and $x_i^{kr} \in \text{in}_{<}(J_{n,k})$.
- If $S \subseteq [n]$ satisfies $|S| = n - k + 1$, we also have $\mathbf{x}(S)^r \in \text{in}_{<}(I_{n,k})$ and $\mathbf{x}(S)^r \in \text{in}_{<}(J_{n,k})$.

Proof. The first claim follows from the fact that x_i^{kr+1} is a generator of $I_{n,k}$ and x_i^{kr} is a generator of $J_{n,k}$. The second claim is a consequence of Lemma 3.3 and Lemma 3.4. \square

It will turn out that the monomials given in Lemma 6.5 will suffice to generate $\text{in}_{<}(I_{n,k})$ and $\text{in}_{<}(J_{n,k})$. The next definition gives the family of monomials which are not divisible by any of the monomials in Lemma 6.5 a name.

Definition 4.4. A monomial $m \in \mathbb{C}[\mathbf{x}_n]$ is (n, k) -nonskip if

- $x_i^{kr+1} \nmid m$ for $1 \leq i \leq n$, and
- $\mathbf{x}(S)^r \nmid m$ for all $S \subseteq [n]$ with $|S| = n - k + 1$.

Let $\mathcal{M}_{n,k}$ denote the collection of all (n, k, r) -nonskip monomials in $\mathbb{C}[\mathbf{x}_n]$.

An (n, k) -nonskip monomial $m \in \mathcal{M}_{n,k}$ is called *strongly* (n, k) -nonskip if we have $x_i^{kr} \nmid m$ for all $1 \leq i \leq n$. Let $\mathcal{N}_{n,k}$ denote the collection of strongly (n, k) -nonskip monomials.

We will describe a bijection $\Psi : \mathcal{F}_{n,k} \rightarrow \mathcal{M}_{n,k}$ which restricts to a bijection $\mathcal{OP}_{n,k} \rightarrow \mathcal{N}_{n,k}$. The bijection Ψ will be constructed recursively, so that $\Psi(\sigma)$ will be determined by $\Psi(\bar{\sigma})$, where $\bar{\sigma}$ is the G_{n-1} -face obtained from σ by deleting the largest letter n . The recursive procedure which gives the derivation $\Psi(\bar{\sigma}) \mapsto \Psi(\sigma)$ will rely on the following lemmata involving skip monomials. The first of these is an extension of [14, Lem. 4.5].

Lemma 4.5. *Let $m \in \mathbb{C}[\mathbf{x}_n]$ be a monomial and let $S, T \subseteq [n]$ be subsets. If $\mathbf{x}(S)^r \mid m$ and $\mathbf{x}(T)^r \mid m$, then $\mathbf{x}(S \cup T)^r \mid m$.*

Proof. Given $i \in S$, it follows from the definition of skip monomials that the exponent of x_i in $\mathbf{x}(S \cup T)^r$ is \leq the exponent of x_i in $\mathbf{x}(S)^r$. A similar observation holds for $i \in T$. The claimed divisibility follows. \square

The following result is an immediate consequence of Lemma 4.5; it extends [14, Lem. 4.6].

Lemma 4.6. *Let $m \in \mathbb{C}[\mathbf{x}_n]$ be a monomial and let ℓ be the largest integer such that there exists a subset $S \subseteq [n]$ with $|S| = \ell$ and $\mathbf{x}(S)^r \mid m$. Then there exists a unique subset $S \subseteq [n]$ with $|S| = \ell$ and $\mathbf{x}(S)^r \mid m$.*

Proof. If there were two such sets S, S' then by Lemma 4.5 we would have $\mathbf{x}(S \cup S')^r \mid m$, contradicting the definition of ℓ . \square

Given any subset $S \subseteq [n]$, let $\mathbf{m}(S) := \prod_{i \in S} x_i$ be the corresponding squarefree monomial. For example, we have $\mathbf{m}(245) = x_2 x_4 x_5$. We have the following lemma involving the r^{th} power $\mathbf{m}(S)^r$ of $\mathbf{m}(S)$. This is the extension of [14, Lem. 4.7].

Lemma 4.7. *Let $m \in \mathcal{M}_{n,k}$ be an (n, k) -nonskip monomial. There exists a unique set $S \subseteq [n]$ with $|S| = n - k$ such that*

- (1) $\mathbf{x}(S)^r \mid (\mathbf{m}(S)^r \cdot m)$, and
- (2) $\mathbf{x}(U)^r \nmid (\mathbf{m}(S)^r \cdot m)$ for all $U \subseteq [n]$ with $|U| = n - k + 1$.

Proof. We begin with uniqueness. Suppose $S = \{s_1 < \dots < s_{n-k}\}$ and $T = \{t_1 < \dots < t_{n-k}\}$ were two such sets. Let ℓ be such that $s_1 = t_1, \dots, s_{\ell-1} = t_{\ell-1}$, and $s_\ell \neq t_\ell$; without loss of generality we have $s_\ell < t_\ell$. Define a new set U by $U := \{s_1 < \dots < s_\ell < t_\ell < t_{\ell+1} < \dots < t_{n-k}\}$, so that $|U| = n - k + 1$. Since $\mathbf{x}(S)^r \mid (\mathbf{m}(S)^r \cdot m)$ and $\mathbf{x}(T)^r \mid (\mathbf{m}(T)^r \cdot m)$, we have $\mathbf{x}(U)^r \mid (\mathbf{m}(S)^r \cdot m)$, which is a contradiction.

To prove existence, consider the following collection \mathcal{C} of subsets of $[n]$:

$$(4.7) \quad \mathcal{C} := \{S \subseteq [n] : |S| = n - k \text{ and } \mathbf{x}(S)^r \mid (\mathbf{m}(S)^r \cdot m)\}.$$

The collection \mathcal{C} is nonempty; indeed, we have $\{1, 2, \dots, n - k\} \in \mathcal{C}$. Let $S_0 \in \mathcal{C}$ be the lexicographically *final* set in \mathcal{C} ; we argue that $\mathbf{m}(S_0)^r \cdot m$ satisfies Condition 2 of the statement of the lemma, thus finishing the proof.

Let $U \subseteq [n]$ have size $|U| = n - k + 1$ and suppose $\mathbf{x}(U)^r \mid (\mathbf{m}(S_0)^r \cdot m)$. If there were an element $u \in U$ with $u < \min(S_0)$, then we would have $\mathbf{x}(S_0 \cup \{u\})^r \mid m$, which contradicts the assumption $m \in \mathcal{M}_{n,k}$. Since $|U| > |S_0|$, there exists an element $u_0 \in U - S_0$ with $u_0 > \min(S_0)$. Write the union $S_0 \cup \{u_0\}$ as

$$(4.8) \quad S_0 \cup \{u_0\} = \{s_1 < \dots < s_j < u_0 < s_{j+1} < \dots < s_{n-k}\},$$

where $j \geq 1$. Define a new set S'_0 by

$$(4.9) \quad S'_0 := \{s_1 < \cdots < s_{j-1} < u_0 < s_{j+1} < \cdots < s_{n-k}\}.$$

Then S'_0 comes after S_0 in lexicographic order but we have $S'_0 \in \mathcal{C}$, contradicting our choice of S_0 . \square

To see how Lemma 4.7 works, consider the case $(n, k, r) = (5, 2, 3)$ and $m = x_1^2 x_2^6 x_3^3 x_4^3 x_5^6 \in \mathcal{M}_{5,2}$. The collection \mathcal{C} of sets

$$\mathcal{C} = \{S \subseteq [5] : |S| = 3 \text{ and } \mathbf{x}(S)^3 \mid (\mathbf{m}(S)^3 \cdot m)\}$$

is given by

$$\mathcal{C} = \{123, 124, 125, 234, 235\}.$$

However, we have

$$\begin{aligned} \mathbf{x}(1234)^3 \mid (\mathbf{m}(123)^3 \cdot m), & \quad \mathbf{x}(1234)^3 \mid (\mathbf{m}(124)^3 \cdot m), \\ \mathbf{x}(1235)^3 \mid (\mathbf{m}(125)^3 \cdot m), & \quad \mathbf{x}(2345)^3 \mid (\mathbf{m}(234)^3 \cdot m). \end{aligned}$$

On the other hand, if $S \subseteq [5]$ and $|S| = 4$, then $\mathbf{x}(S)^3 \nmid (\mathbf{m}(S)^3 \cdot m)$. Observe that 235 is the lexicographically final set in \mathcal{C} .

4.3. The bijection Ψ . We describe a bijection $\Psi : \mathcal{F}_{n,k} \rightarrow \mathcal{M}_{n,k}$ which restricts to a bijection $\mathcal{OP}_{n,k} \rightarrow \mathcal{N}_{n,k}$ with the property that $\text{coinv}(\sigma) = \deg(\Psi(\sigma))$ for any G_n -face $\sigma \in \mathcal{F}_{n,k}$. The construction of Ψ will be recursive in the parameter n .

If $n = 1$ and $k = 1$, the relation $\text{coinv}(\sigma) = \deg(\Psi(\sigma))$ determines the bijection Ψ uniquely. Explicitly, the map $\Psi : \mathcal{F}_{1,1} \rightarrow \mathcal{M}_{1,1}$ is defined by

$$(4.10) \quad \Psi : (1^c) \mapsto x_1^{r-c-1},$$

for any color $0 \leq c \leq r - 1$.

If $n = 1$ and $k = 0$ then $\mathcal{F}_{1,0}$ consists of the sole face (1) . On the other hand, the collection $\mathcal{M}_{1,0}$ of nonskip monomials consists of the sole monomial 1. We are forced to define

$$(4.11) \quad \Psi : (1) \mapsto 1.$$

The combinatorial recursion on which Ψ is based is as follows. Let $\sigma = (B_1 \mid \cdots \mid B_\ell) \in \mathcal{F}_{n,k}$ be an G_n -face of dimension k , so that $\ell = k + 1$ or $\ell = k$ according to whether σ has a zero block. Suppose we wish to build a larger face by inserting $n + 1$ into σ . There are three ways in which this can be done.

- (1) We could perform a *star insertion* by inserting $n + 1$ into one of the nonzero blocks $B_{\ell-j}$ of σ for $1 \leq j \leq k$ also assigning a color c to $n + 1$. The resulting G_n -face would be $(B_1 \mid \cdots \mid B_{\ell-j} \cup \{(n+1)^c\} \mid \cdots \mid B_\ell)$. This leaves the dimension k unchanged and increases coinv by $r \cdot (k - j) + (r - c - 1)$.

For example, if $r = 2$ and $\sigma = (3 \mid 2^1 4^0 \mid 1^1) \in \mathcal{F}_{4,2}$, the possible star insertions of 5 and their effects on coinv are

$$\begin{array}{cccc} (3 \mid 2^1 4^0 5^1 \mid 1^1) & (3 \mid 2^1 4^0 5^0 \mid 1^1) & (3 \mid 2^1 4^0 \mid 1^1 5^1) & (3 \mid 2^1 4^0 \mid 1^1 5^0) \\ \text{coinv} + 0 & \text{coinv} + 1 & \text{coinv} + 2 & \text{coinv} + 3. \end{array}$$

- (2) We could perform a *zero insertion* by inserting $n + 1$ into the zero block of σ (or by creating a new zero block whose sole element is $n + 1$). This leaves the dimension k unchanged and increases coinv by kr .

For example, if $r = 2$ and $\sigma = (3 \mid 2^1 4^0 \mid 1^1) \in \mathcal{F}_{4,2}$, the zero insertion of 5 would yield $(35 \mid 2^1 4^0 \mid 1^1)$, adding 4 to coinv .

- (3) We could perform a *bar insertion* by inserting $n + 1$ into a new singleton nonzero block of σ just after the block $B_{\ell-j}$ for some $0 \leq j \leq k$, also assigning a color c to $n + 1$. The resulting G_n -face would be $(B_1 \mid \cdots \mid B_{\ell-j} \mid (n+1)^c \mid B_{\ell-j+1} \mid \cdots \mid B_\ell)$. This increases the dimension k by one and increases coinv by $r \cdot (n - k) + r \cdot (k - j) + (r - c - 1)$.

For example, if $r = 2$ and $\sigma = (3 \mid 2^1 4^0 \mid 1^1) \in \mathcal{F}_{4,2}$, the possible bar insertions of 5 and their effects on coinv are

$$\begin{array}{ccc} (3 \mid 5^1 \mid 2^1 4^0 \mid 1^1) & (3 \mid 5^0 \mid 2^1 4^0 \mid 1^1) & (3 \mid 2^1 4^0 \mid 5^1 \mid 1^1) \\ \text{coinv} + 4 & \text{coinv} + 5 & \text{coinv} + 6 \\ \\ (3 \mid 2^1 4^0 \mid 5^0 \mid 1^1) & (3 \mid 2^1 4^0 \mid 1^1 \mid 5^1) & (3 \mid 2^1 4^0 \mid 1^1 \mid 5^0) \\ \text{coinv} + 7 & \text{coinv} + 8 & \text{coinv} + 9. \end{array}$$

The names of these three kinds of insertions come from our combinatorial models for G_n -faces; a star insertion adds a star to the star model of σ , a zero insertion adds an element to the zero block of σ , and a bar insertion adds a bar to the bar model of σ .

Let $\sigma = (B_1 \mid \cdots \mid B_\ell) \in \mathcal{F}_{n,k}$ be an G_n -face of dimension k and let $\bar{\sigma}$ be the G_{n-1} -face obtained by deleting n from σ . Then $\bar{\sigma} \in \mathcal{F}_{n-1,k}$ if σ arises from $\bar{\sigma}$ by a star or zero insertion and $\bar{\sigma} \in \mathcal{F}_{n-1,k-1}$ if σ arises from $\bar{\sigma}$ from a bar insertion. Assume inductively that the monomial $\Psi(\bar{\sigma})$ has been defined, and that this monomial lies in $\mathcal{M}_{n-1,k}$ or $\mathcal{M}_{n-1,k-1}$ according to whether $\bar{\sigma}$ lies in $\mathcal{F}_{n-1,k}$ or $\mathcal{F}_{n-1,k-1}$. We define $\Psi(\sigma)$ by the rule

$$(4.12) \quad \Psi(\sigma) := \begin{cases} \Psi(\bar{\sigma}) \cdot x_n^{r \cdot (k-j-1) + (r-c-1)} & \text{if } n^c \in B_{\ell-j} \text{ and } B_{\ell-j} \text{ is a nonzero nonsingleton,} \\ \Psi(\bar{\sigma}) \cdot x_n^{kr} & \text{if } n \text{ lies in the zero block of } \sigma, \\ \Psi(\bar{\sigma}) \cdot \mathbf{m}(S)^r \cdot x_n^{r \cdot (k-j-1) + (r-c-1)} & \text{if } B_{\ell-j} = \{n^c\} \text{ is a nonzero singleton,} \end{cases}$$

where in the third branch $S \subseteq [n-1]$ is the unique subset of size $|S| = n - k$ guaranteed by Lemma 4.7 applied to $m = \Psi(\bar{\sigma}) \in \mathcal{M}_{n-1,k-1}$.

Example 4.8. Let $(n, k, r) = (8, 3, 3)$ and consider the face $\sigma = (25 \mid 1^0 7^0 8^1 \mid 6^1 \mid 3^2 4^2) \in \mathcal{F}_{8,3}$. In order to calculate $\Psi(\sigma) \in \mathcal{M}_{8,3}$, we refer to the following table. Here ‘type’ refers to the type of insertion (star, zero, or bar) of n at each stage.

| σ | n | k | type | S | $\Psi(\sigma)$ |
|---|-----|-----|------|-----|---|
| (1^0) | 1 | 1 | | | x_1^2 |
| $(2 \mid 1^0)$ | 2 | 1 | zero | | $x_1^2 x_2^3$ |
| $(2 \mid 1^0 \mid 3^2)$ | 3 | 2 | bar | 2 | $x_1^2 x_2^3 \cdot \mathbf{m}(2)^3 \cdot x_3^3 = x_1^2 x_2^6 x_3^3$ |
| $(2 \mid 1^0 \mid 3^2 4^2)$ | 4 | 2 | star | | $x_1^2 x_2^6 x_3^3 x_4^3$ |
| $(25 \mid 1^0 \mid 3^2 4^2)$ | 5 | 2 | zero | | $x_1^2 x_2^6 x_3^3 x_4^3 x_5^6$ |
| $(25 \mid 1^0 \mid 6^1 \mid 3^2 4^2)$ | 6 | 3 | bar | 235 | $x_1^2 x_2^6 x_3^3 x_4^3 x_5^6 \cdot \mathbf{m}(235)^3 \cdot x_6^4 = x_1^2 x_2^9 x_3^6 x_4^3 x_5^9 x_6^4$ |
| $(25 \mid 1^0 7^0 \mid 6^1 \mid 3^2 4^2)$ | 7 | 3 | star | | $x_1^2 x_2^9 x_3^6 x_4^3 x_5^9 x_6^4 x_7^2$ |
| $(25 \mid 1^0 7^0 8^1 \mid 6^1 \mid 3^2 4^2)$ | 8 | 3 | star | | $x_1^2 x_2^9 x_3^6 x_4^3 x_5^9 x_6^4 x_7^2 x_8^1$ |

We conclude that

$$\Psi(\sigma) = \Psi(25 \mid 1^0 7^0 8^1 \mid 6^1 \mid 3^2 4^2) = x_1^2 x_2^9 x_3^6 x_4^3 x_5^9 x_6^4 x_7^2 x_8^1 \in \mathcal{M}_{8,3}.$$

Observe that the zero block of σ is $\{2, 5\}$, and that x_2 and x_5 are the variables in $\Psi(\sigma)$ with exponent $kr = 3 \cdot 3 = 9$.

The next result is the extension of [14, Thm. 4.9] to $r \geq 2$. The proof has the same basic structure, but one must account for the presence of zero blocks.

Proposition 4.9. *The map $\Psi : \mathcal{F}_{n,k} \rightarrow \mathcal{M}_{n,k}$ is a bijection which restricts to a bijection $\mathcal{OP}_{n,k} \rightarrow \mathcal{N}_{n,k}$. Moreover, for any $\sigma \in \mathcal{F}_{n,k}$ we have*

$$(4.13) \quad \text{coinv}(\sigma) = \deg(\Psi(\sigma)).$$

Finally, if $\sigma \in \mathcal{F}_{n,k}$ has a zero block Z , then

$$(4.14) \quad Z = \{1 \leq i \leq n : \text{the exponent of } x_i \text{ in } \Psi(\sigma) \text{ is } kr\}.$$

Proof. We need to show that Ψ is a well-defined function $\mathcal{F}_{n,k} \rightarrow \mathcal{M}_{n,k}$. To do this, we induct on n (with the base case $n = 1$ being clear). Let $\sigma = (B_1 \mid \cdots \mid B_\ell) \in \mathcal{F}_{n,k}$ and let $\bar{\sigma}$ be the G_{n-1} -face obtained by removing n from σ . Then $\bar{\sigma} \in \mathcal{F}_{n-1,k}$ (if the insertion type of n was star or zero) or $\bar{\sigma} \in \mathcal{F}_{n-1,k-1}$ (if the insertion type of n was bar). We inductively assume that $\Psi(\bar{\sigma}) \in \mathcal{M}_{n-1,k}$ or $\Psi(\bar{\sigma}) \in \mathcal{M}_{n-1,k-1}$ accordingly.

Suppose first that the insertion type of n was star or zero, so that $\Psi(\bar{\sigma}) \in \mathcal{M}_{n-1,k}$. Then we have

$$(4.15) \quad \Psi(\sigma) = \begin{cases} \Psi(\bar{\sigma}) \cdot x_n^{r \cdot (k-j-1) + (r-c-1)} & \text{if } n^c \in B_{\ell-j} \text{ and } B_{\ell-j} \text{ is a nonzero nonsingleton,} \\ \Psi(\bar{\sigma}) \cdot x_n^{kr} & \text{if } n \text{ lies in the zero block of } \sigma. \end{cases}$$

By induction and the inequalities $0 \leq j \leq k-1$ and $0 \leq c \leq r-1$, we know that none of the variable powers $x_1^{kr+1}, \dots, x_n^{kr+1}$ divide $\Psi(\sigma)$. Let $S \subseteq [n]$ be a subset of size $|S| = n - k + 1$. Since $\Psi(\bar{\sigma}) \in \mathcal{M}_{n-1,k}^r$, we know that $\mathbf{x}(S - \{\max(S)\})^r \nmid \Psi(\bar{\sigma})$. This implies that $\mathbf{x}(S)^r \nmid \Psi(\sigma)$. We conclude that $\Psi(\sigma) \in \mathcal{M}_{n,k}$.

Now suppose that the insertion type of n was bar, so that $\Psi(\bar{\sigma}) \in \mathcal{M}_{n-1,k-1}$. We have

$$(4.16) \quad \Psi(\sigma) = \Psi(\bar{\sigma}) \cdot \mathbf{m}(S)^r \cdot x_n^{r \cdot (k-j-1) + (r-c-1)},$$

where $B_{\ell-j} = \{n^c\}$ and $S \subseteq [n-1]$ is the unique subset of size $|S| = n - k$ guaranteed by Lemma 4.7 applied to the monomial $m = \Psi(\bar{\sigma})$. Since none of the variable powers $x_1^{(k-1) \cdot r + 1}, \dots, x_{n-1}^{(k-1) \cdot r + 1}$ divide $\Psi(\bar{\sigma})$, we conclude that none of the variable powers $x_1^{kr+1}, \dots, x_n^{kr+1}$ divide $\Psi(\sigma)$. Let $T \subseteq [n]$ satisfy $|T| = n - k + 1$. If $n \notin T$, Lemma 4.7 and induction guarantee that $\mathbf{x}(T)^r \nmid \Psi(\sigma)$. If $n \in T$, then the power of x_n in the monomial $\mathbf{x}(T)^r$ is kr , so that $\mathbf{x}(T)^r \nmid \Psi(\sigma)$. We conclude that $\Psi(\sigma) \in \mathcal{M}_{n,k}$. This finishes the proof that $\Psi : \mathcal{F}_{n,k} \rightarrow \mathcal{M}_{n,k}$ is well-defined.

The relationship $\text{coinv}(\sigma) = \deg(\Psi(\sigma))$ is clear from the inductive definition of Ψ and the previously described effect of insertion on the coinv statistic.

Let $\sigma \in \mathcal{F}_{n,k}$ be an G_n -face with zero block Z (where Z could be empty). We aim to show that $Z = \{1 \leq i \leq n : \text{the exponent of } x_i \text{ in } \Psi(\sigma) \text{ is } kr\}$. To do this, we proceed by induction on n (the case $n = 1$ being clear). As before, let $\bar{\sigma}$ be the face obtained by erasing n from σ and let \bar{Z} be the zero block of $\bar{\sigma}$. We inductively assume that

$$(4.17) \quad \bar{Z} = \begin{cases} \{1 \leq i \leq n-1 : \text{the exponent of } x_i \text{ in } \Psi(\bar{\sigma}) \text{ is } kr\} & \text{if } \bar{\sigma} \in \mathcal{F}_{n-1,k}, \\ \{1 \leq i \leq n-1 : \text{the exponent of } x_i \text{ in } \Psi(\bar{\sigma}) \text{ is } (k-1) \cdot r\} & \text{if } \bar{\sigma} \in \mathcal{F}_{n-1,k-1}. \end{cases}$$

Suppose first that σ was obtained from $\bar{\sigma}$ by a star insertion, so that $\bar{\sigma} \in \mathcal{F}_{n-1,k}$ and $Z = \bar{Z}$. Since the exponent of x_n in $\Psi(\sigma)$ is $< kr$, the desired equality of sets holds in this case.

Next, suppose that σ was obtained from $\bar{\sigma}$ by a zero insertion, so that $\bar{\sigma} \in \mathcal{F}_{n-1,k}$ and $Z = \bar{Z} \cup \{n\}$. Since the exponent of x_n in $\Psi(\sigma)$ is kr , the desired equality of sets holds in this case.

Finally, suppose that σ was obtained from $\bar{\sigma}$ by a bar insertion, so that $\bar{\sigma} \in \mathcal{F}_{n-1,k-1}$ and $Z = \bar{Z}$. Since the exponent of x_n in $\Psi(\sigma)$ is $< kr$, by induction we need only argue that $Z \subseteq S$, where $S \subseteq [n-1]$ is the unique subset of size $|S| = n - k$ guaranteed by Lemma 4.7 applied to the monomial $m = \Psi(\bar{\sigma})$.

If the containment $Z \subseteq S$ failed to hold, let $z = Z - S$ be arbitrary. By induction, the exponent of x_z in $\Psi(\bar{\sigma})$ is $(k-1) \cdot r$. Also, we have the divisibility $\mathbf{x}(S)^r \mid \Psi(\bar{\sigma}) \cdot \mathbf{m}(S)^r$. If since $z \leq n-1$, we have the divisibility $\mathbf{x}(S \cup \{z\})^r \mid \mathbf{x}(S)^r \cdot x_z^{(k-1) \cdot r}$, so that $\mathbf{x}(S \cup \{z\})^r \mid \Psi(\bar{\sigma}) \cdot \mathbf{m}(S)^r$, which contradicts Lemma 4.7. We conclude that $Z \subseteq S$. This proves the last sentence of the proposition.

We now turn our attention to proving that $\Psi : \mathcal{F}_{n,k} \rightarrow \mathcal{M}_{n,k}$ is a bijection. In order to prove that Ψ is a bijection, we will construct its inverse $\Phi : \mathcal{M}_{n,k} \rightarrow \mathcal{F}_{n,k}$. The map Φ will be defined by reversing the recursion used to define Ψ .

When $(n, k) = (1, 0)$, there is only one choice for Φ ; we must define $\Phi : \mathcal{M}_{1,0} \rightarrow \mathcal{F}_{1,0}$ by

$$(4.18) \quad \Phi : 1 \mapsto (1).$$

When $(n, k) = (1, 1)$, since Φ is supposed to invert the function Ψ , we are forced to define $\Phi : \mathcal{M}_{1,1} \rightarrow \mathcal{F}_{1,1}$ by

$$(4.19) \quad \Phi : x_1^c \mapsto (1^{r-c-1}),$$

for $0 \leq c \leq r-1$.

In general, fix $k \leq n$ and assume inductively that the functions

$$\begin{cases} \Phi : \mathcal{M}_{n-1,k} \rightarrow \mathcal{F}_{n-1,k}, \\ \Phi : \mathcal{M}_{n-1,k-1} \rightarrow \mathcal{F}_{n-1,k-1} \end{cases}$$

have already been defined. We aim to define the function $\Phi : \mathcal{M}_{n,k} \rightarrow \mathcal{F}_{n,k}$. To this end, let $m = x_1^{a_1} \cdots x_{n-1}^{a_{n-1}} x_n^{a_n} \in \mathcal{M}_{n,k}$ be a monomial. Define a new monomial $m' := x_1^{a_1} \cdots x_{n-1}^{a_{n-1}}$ by setting $x_n = 1$ in m . Either $m' \in \mathcal{M}_{n-1,k}$ or $m' \notin \mathcal{M}_{n-1,k}$.

If $m' \in \mathcal{M}_{n-1,k}$, then $\Phi(m') = (B_1 \mid \cdots \mid B_\ell) \in \mathcal{F}_{n-1,k}^r$ is a previously defined G_{n-1} -face. Our definition of $\Phi(m)$ depends on the exponent a_n of x_n in m .

- If $m' \in \mathcal{M}_{n-1,k}$ and $a_n < kr$, write $a_n = j \cdot r + (r - c - 1)$ for a nonnegative integer j and $0 \leq c \leq r - 1$. Let $\Phi(m)$ be obtained from $\Phi(m')$ by star inserting n^c into the j^{th} nonzero block of $\Psi(m)$ from the left.
- If $m' \in \mathcal{M}_{n-1,k}$ and $a_n = kr$, let $\Phi(m)$ be obtained from $\Phi(m')$ by adding n to the zero block of $\Phi(m')$ (creating a zero block if necessary).

If $m' \notin \mathcal{M}_{n-1,k}$, there exists a subset $S \subseteq [n-1]$ such that $|S| = n-k$ and $\mathbf{x}(S)^r \mid m'$. Lemma 4.6 guarantees that the set S is unique.

Claim: We have $\frac{m'}{\mathbf{m}(S)^r} \in \mathcal{M}_{n-1,k-1}$.

Since $m \in \mathcal{M}_{n,k}$, we know that $\mathbf{x}(T)^r \nmid \frac{m'}{\mathbf{m}(S)^r}$ for all $T \subseteq [n-1]$ with $|T| = n-k+1$. Let $1 \leq j \leq n-1$. We need to show $x_j^{(k-1)r+1} \nmid \frac{m'}{\mathbf{m}(S)^r}$. If $j \in S$ this is immediate from the fact that $x_j^{kr+1} \nmid m'$. If $j \notin S$ and $x_j^{(k-1)r+1} \mid \frac{m'}{\mathbf{m}(S)^r}$, then $x_j^{(k-1)r+1} \mid m'$ and $\mathbf{x}(S \cup \{j\})^r \mid m'$, a contradiction to the assumption $m = m' \cdot x_n^{a_n} \in \mathcal{M}_{n,k}$. This finishes the proof of the Claim.

By the Claim, we recursively have an G_{n-1} -face $\Phi\left(\frac{m'}{\mathbf{m}(S)^r}\right) \in \mathcal{F}_{n-1,k-1}$. Moreover, we have $a_n < kr$ (because otherwise $\mathbf{x}(S \cup \{n\})^r \mid m$, contradicting $m \in \mathcal{M}_{n,k}$). Write $a_n = j \cdot r + (r - c - 1)$ for some nonnegative integer j and $0 \leq c \leq r - 1$. Form $\Phi(m)$ from $\Phi(m')$ by bar inserting the singleton block $\{n^c\}$ to the left of the j^{th} nonzero block of $\Phi(m')$ from the left.

For an example of the map Φ , let $(n, k, r) = (8, 3, 3)$ and let $m = x_1^2 x_2^9 x_3^6 x_4^3 x_5^9 x_6^4 x_7^2 x_8^1 \in \mathcal{M}_{8,3}$. The following table computes $\Phi(m) = (25 \mid 1^0 7^0 8^1 \mid 6^1 \mid 3^2 4^2)$. Throughout this calculation, the nonzero blocks will successively become frozen (i.e., written in bold).

| m | m' | (n, k) | type | S | $\frac{m'}{\mathbf{m}(S)^r}$ | (j, c) | $\Phi(m)$ |
|---|---|----------|------|-------------|---------------------------------|----------|--|
| $x_1^2 x_2^9 x_3^6 x_4^3 x_5^9 x_6^4 x_7^2 x_8^1$ | $x_1^2 x_2^9 x_3^6 x_4^3 x_5^9 x_6^4 x_7^2$ | $(8, 3)$ | star | | | $(0, 1)$ | $(8^1 \mid \cdot \mid \cdot)$ |
| $x_1^2 x_2^9 x_3^6 x_4^3 x_5^9 x_6^4 x_7^2$ | $x_1^2 x_2^9 x_3^6 x_4^3 x_5^9 x_6^4$ | $(7, 3)$ | star | | | $(0, 0)$ | $(7^0 8^1 \mid \cdot \mid \cdot)$ |
| $x_1^2 x_2^9 x_3^6 x_4^3 x_5^9 x_6^4$ | $x_1^2 x_2^9 x_3^6 x_4^3 x_5^9$ | $(6, 3)$ | bar | 235 | $x_1^2 x_2^6 x_3^3 x_4^3 x_5^6$ | $(1, 1)$ | $(7^0 8^1 \mid \mathbf{6}^1 \mid \cdot)$ |
| $x_1^2 x_2^6 x_3^3 x_4^3 x_5^6$ | $x_1^2 x_2^6 x_3^3 x_4^3$ | $(5, 2)$ | zero | | | | $(5 \mid 7^0 8^1 \mid \mathbf{6}^1 \mid \cdot)$ |
| $x_1^2 x_2^6 x_3^3 x_4^3$ | $x_1^2 x_2^6 x_3^3$ | $(4, 2)$ | star | | | $(1, 2)$ | $(5 \mid 7^0 8^1 \mid \mathbf{6}^1 \mid 4^2)$ |
| $x_1^2 x_2^6 x_3^3$ | $x_1^2 x_2^6$ | $(3, 2)$ | bar | 2 | $x_1^2 x_2^3$ | $(1, 2)$ | $(5 \mid 7^0 8^1 \mid \mathbf{6}^1 \mid \mathbf{3}^2 4^2)$ |
| $x_1^2 x_2^3$ | x_1^2 | $(2, 1)$ | zero | | | | $(25 \mid 7^0 8^1 \mid \mathbf{6}^1 \mid \mathbf{3}^2 4^2)$ |
| x_1^2 | 1 | $(1, 1)$ | bar | \emptyset | 1 | $(0, 0)$ | $(25 \mid \mathbf{1}^0 7^0 8^1 \mid \mathbf{6}^1 \mid \mathbf{3}^2 4^2)$ |

To proceed from one row of the table to the next, we use the following procedure.

- Define m to be the monomial m' from the above row (if the insertion type in the above row was star or zero) or the monomial $\frac{m'}{\mathbf{m}(S)^r}$ from the above row (if the insertion type in the above row was bar).
- Define (n, k) in the current row to be $(n - 1, k)$ from the above row (if the insertion type in the above row was star or zero) or $(n - 1, k - 1)$ from the above row (if the insertion type in the above row was bar).
- Using the (n, k) in the current row, define m' from m using the relation $m = m' \cdot x_n^{a_n}$.
- If $a_n = kr$, define the insertion type of the current row to be zero, let $\Phi(m)$ be obtained from the above row by adjoining n to its zero block (creating a new zero block if necessary), and move on to the next row.
- If $a_n < kr$, define (j, c) by the relation $a_n = j \cdot r + (r - c - 1)$, where j is nonnegative and $0 \leq c \leq r - 1$.
- If $a_n < kr$ and $m' \in \mathcal{M}_{n-1, k}$, define the insertion type of the current row to be star. Let $\Phi(m)$ obtained from the above row by inserting n^c into the j^{th} nonzero nonfrozen block from the left, and move on to the next row.
- If $a_n < kr$ and $m' \notin \mathcal{M}_{n-1, k}$, define the insertion type of the current row to be bar. Let $S \subseteq [n - 1]$ be the set defined by Lemma 4.6 as above. Calculate $\frac{m'}{\mathbf{m}(S)^r}$. Let $\Phi(m)$ be obtained from the above row by inserting n^c into the j^{th} nonzero nonfrozen block from the left and freezing that block. Move on to the next row.

We leave it for the reader to check that the procedure defined above reverses the recursive definition of Ψ , so that Φ and Ψ are mutually inverse maps. The fact that Ψ restricts to give a bijection $\mathcal{OP}_{n, k} \rightarrow \mathcal{N}_{n, k}$ follows from the assertion about zero blocks. \square

We are ready to identify the standard monomial bases of our quotient rings $R_{n, k}$ and $S_{n, k}$. The proof of the following result is analogous to the proof of [14, Thm. 4.10].

Theorem 4.10. *Let $n \geq k$ be positive integers and endow monomials in $\mathbb{C}[\mathbf{x}_n]$ with the lexicographic term order $<$.*

- *The collection $\mathcal{M}_{n, k}$ of (n, k) -nonskip monomials in $\mathbb{C}[\mathbf{x}_n]$ is the standard monomial basis of $R_{n, k}$.*
- *The collection $\mathcal{N}_{n, k}$ of strongly (n, k) -nonskip monomials in $\mathbb{C}[\mathbf{x}_n]$ is the standard monomial basis of $S_{n, k}$.*

Proof. Let us begin with the case of $R_{n, k}$. Recall the point set $Y_{n, k} \subseteq \mathbb{C}^n$. Let $\mathcal{B}_{n, k}$ be the standard monomial basis of the quotient ring $\mathbb{C}[\mathbf{x}_n]/\mathbf{T}(Y_{n, k})$. Since $\dim(\mathbb{C}[\mathbf{x}_n]/\mathbf{T}(Y_{n, k})) = |Y_{n, k}| = |\mathcal{F}_{n, k}|$, we have

$$(4.20) \quad |\mathcal{B}_{n, k}| = |\mathcal{F}_{n, k}|.$$

On the other hand, Lemma 4.2 says that $I_{n,k} \subseteq \mathbf{T}(Y_{n,k})$. This leads to the containment of initial ideals

$$(4.21) \quad \text{in}_<(I_{n,k}) \subseteq \text{in}_<(\mathbf{T}(Y_{n,k})).$$

If $\mathcal{C}_{n,k}$ is the standard monomial basis for $R_{n,k} = \mathbb{C}[\mathbf{x}_n]/I_{n,k}$, this implies

$$(4.22) \quad \mathcal{B}_{n,k} \subseteq \mathcal{C}_{n,k}.$$

However, Lemma 6.5 and the definition of (n, k) -nonskip monomials implies

$$(4.23) \quad \mathcal{C}_{n,k} \subseteq \mathcal{M}_{n,k}.$$

Proposition 4.9 shows that $|\mathcal{M}_{n,k}| = |\mathcal{F}_{n,k}|$. Since we already know $\mathcal{B}_{n,k} \subseteq \mathcal{M}_{n,k}$ and $|\mathcal{B}_{n,k}| = |\mathcal{F}_{n,k}|$, we conclude that

$$(4.24) \quad \mathcal{B}_{n,k} = \mathcal{M}_{n,k},$$

which proves the first assertion of the theorem.

The case of $S_{n,k}$ is similar. An identical chain of reasoning, this time involving $Z_{n,k}$ instead of $Y_{n,k}$, shows that $\mathcal{N}_{n,k}$ contains the standard monomial basis for $S_{n,k}$. Proposition 4.9 implies that both $|\mathcal{N}_{n,k}|$ and $\dim(S_{n,k})$ equal $|\mathcal{OP}_{n,k}|$. \square

Theorem 4.10 makes it easy to compute the Hilbert series of $R_{n,k}$ and $S_{n,k}$.

Corollary 4.11. *The graded vector spaces $R_{n,k}$ and $S_{n,k}$ have the following Hilbert series.*

$$(4.25) \quad \text{Hilb}(R_{n,k}; q) = \sum_{z=0}^n \binom{n}{z} q^{krz} \cdot \text{rev}_q([r]_q^{n-z} \cdot [k]!_{q^r} \cdot \text{Stir}_{q^r}(n-z, k))$$

$$(4.26) \quad = \sum_{z=0}^n \binom{n}{z} q^{krz} \cdot [r]_q^{n-z} \cdot [k]!_{q^r} \cdot \text{rev}_q(\text{Stir}_{q^r}(n-z, k)).$$

$$(4.27) \quad \text{Hilb}(S_{n,k}; q) = \text{rev}_q([r]_q^n \cdot [k]!_{q^r} \cdot \text{Stir}_{q^r}(n, k))$$

$$(4.28) \quad = [r]_q^n \cdot [k]!_{q^r} \cdot \text{rev}_q(\text{Stir}_{q^r}(n, k)).$$

Proof. By Theorem 4.10 and Proposition 4.9, we have

$$(4.29) \quad \text{Hilb}(R_{n,k}; q) = \sum_{\sigma \in \mathcal{F}_{n,k}} q^{\text{coinv}(\sigma)},$$

$$(4.30) \quad \text{Hilb}(S_{n,k}; q) = \sum_{\sigma \in \mathcal{OP}_{n,k}} q^{\text{coinv}(\sigma)},$$

so that the proof of the corollary reduces to calculating the generating function of coinv on $\mathcal{F}_{n,k}$ and $\mathcal{OP}_{n,k}$.

It follows from the work of Steingrímsson [21] that the generating function of coinv on $\mathcal{OP}_{n,k}$ is

$$(4.31) \quad \sum_{\sigma \in \mathcal{OP}_{n,k}} q^{\text{coinv}(\sigma)} = \text{rev}_q([r]_q^n \cdot [k]!_{q^r} \cdot \text{Stir}_{q^r}(n, k)),$$

proving the desired expression for $\text{Hilb}(S_{n,k}; q)$. For the derivation of $\text{Hilb}(R_{n,k}; q)$, simply note that a zero block Z of an G_n -face $\sigma \in \mathcal{F}_{n,k}$ contributes $kr \cdot |Z|$ to $\text{coinv}(\sigma)$. \square

The proof of Theorem 4.10 also gives the *ungraded* isomorphism type of the G_n -modules $R_{n,k}$ and $S_{n,k}$.

Corollary 4.12. *As ungraded G_n -modules we have $R_{n,k} \cong \mathbb{C}[\mathcal{F}_{n,k}]$ and $S_{n,k} \cong \mathbb{C}[\mathcal{OP}_{n,k}]$.*

Proof. We have the following isomorphisms of ungraded G_n -modules:

$$(4.32) \quad \mathbb{C}[\mathbf{x}_n]/\mathbf{T}(Y_{n,k}) \cong \mathbb{C}[\mathbf{x}_n]/I_{n,k} \cong \mathbb{C}[\mathcal{F}_{n,k}]$$

and

$$(4.33) \quad \mathbb{C}[\mathbf{x}_n]/\mathbf{T}(Z_{n,k}) \cong \mathbb{C}[\mathbf{x}_n]/J_{n,k} \cong \mathbb{C}[\mathcal{OP}_{n,k}].$$

The proof of Theorem 4.10 shows that $\mathbf{T}(Y_{n,k}) = I_{n,k}$ and $\mathbf{T}(Z_{n,k}) = J_{n,k}$. \square

Theorem 4.10 identifies the standard monomial bases $\mathcal{M}_{n,k}$ and $\mathcal{N}_{n,k}$ for the quotient rings $R_{n,k}$ and $S_{n,k}$ with respect to the lexicographic term order. However, checking whether monomial $m \in \mathbb{C}[\mathbf{x}_n]$ is (strongly) (n, k) -nonskip involves checking whether $\mathbf{x}(S)^r \mid m$ for all possible subsets $S \subseteq [n]$ with $|S| = n - k + 1$. The next result gives a more direct characterization of the monomials of $\mathcal{M}_{n,k}$ and $\mathcal{N}_{n,k}$.

A *shuffle* of a pair of sequences (a_1, \dots, a_p) and (b_1, \dots, b_q) is an interleaving (c_1, \dots, c_{p+q}) of these sequences which preserves the relative order of the a 's and b 's. The following result is an extension of [14, Thm. 4.13] to $r \geq 2$.

Theorem 4.13. *We have*

$$(4.34) \quad \mathcal{M}_{n,k} = \left\{ x_1^{a_1} \cdots x_n^{a_n} : \begin{array}{l} (a_1, \dots, a_n) \text{ is componentwise } \leq \text{ some shuffle of } \\ (r-1, 2r-1, \dots, kr-1) \text{ and } (kr, \dots, kr) \end{array} \right\},$$

where there are $n - k$ copies of kr . Moreover, we have

$$(4.35) \quad \mathcal{N}_{n,k} = \left\{ x_1^{a_1} \cdots x_n^{a_n} : \begin{array}{l} (a_1, \dots, a_n) \text{ is componentwise } \leq \text{ some shuffle of } \\ (r-1, 2r-1, \dots, kr-1) \text{ and } (kr-1, \dots, kr-1) \end{array} \right\},$$

where there are $n - k$ copies of $kr - 1$.

Proof. Let $\mathcal{A}_{n,k}$ and $\mathcal{B}_{n,k}$ denote the sets of monomials right-hand sides of the top and bottom asserted equalities, respectively. A direct check shows that any shuffle of $(r-1, 2r-1, \dots, kr-1)$ and (kr, \dots, kr) is (n, k) -nonskip and that any shuffle of $(r-1, 2r-1, \dots, kr-1)$ and $(kr-1, \dots, kr-1)$ is (n, k) -strongly nonskip. This implies that $\mathcal{A}_{n,k} \subseteq \mathcal{M}_{n,k}$ and $\mathcal{B}_{n,k} \subseteq \mathcal{N}_{n,k}$.

To verify the reverse containment, consider the bijection $\Psi : \mathcal{F}_{n,k} \rightarrow \mathcal{M}_{n,k}$ of Proposition 4.9. We argue that $\Psi(\mathcal{F}_{n,k}) \subseteq \mathcal{A}_{n,k}$. Let $\sigma \in \mathcal{F}_{n,k}$ be an G_n -face and let $\bar{\sigma}$ be the G_{n-1} -face obtained by removing n from σ .

Case 1: n is not contained in a nonzero singleton block of σ .

In this case we have $\bar{\sigma} \in \mathcal{F}_{n-1,k}$. We inductively assume $\Psi(\bar{\sigma}) \in \mathcal{A}_{n-1,k}$. This means that there is some shuffle (a_1, \dots, a_{n-1}) of the sequences $(r-1, 2r-1, \dots, kr-1)$ and (kr, \dots, kr) such that $\Psi(\bar{\sigma}) \mid x_1^{a_1} \cdots x_{n-1}^{a_{n-1}}$ (where there are $n - k - 1$ copies of kr). By the definition of Ψ we have $\Psi(\sigma) \mid x_1^{a_1} \cdots x_{n-1}^{a_{n-1}} x_n^{kr}$, and $(a_1, \dots, a_{n-1}, kr)$ is a shuffle of $(r-1, 2r-1, \dots, kr-1)$ and (kr, kr, \dots, kr) , where there are $n - k$ copies of kr . We conclude that $\Psi(\sigma) \in \mathcal{A}_{n,k}$.

Case 2: n is contained in a nonzero singleton block of σ .

In this case we have $\bar{\sigma} \in \mathcal{F}_{n-1,k-1}$. We inductively assume $\Psi(\bar{\sigma}) \in \mathcal{A}_{n-1,k-1}$. We have $\Psi(\sigma) = \Psi(\bar{\sigma}) \cdot \mathbf{m}(S)^r \cdot x_n^i$ for some $0 \leq i \leq kr-1$, where $S \subseteq [n-1]$, $|S| = n - k$, and $\mathbf{x}(S)^r \mid (\Psi(\bar{\sigma}) \cdot \mathbf{m}(S)^r)$. Consider the shuffle (a_1, \dots, a_n) of $(r-1, 2r-1, \dots, kr-1)$ and (kr, kr, \dots, kr) determined by $a_j = kr$ if and only if $j \in S$.

We claim $\Psi(\sigma) \mid x_1^{a_1} \cdots x_n^{a_n}$, so that $\Psi(\sigma) \in \mathcal{A}_{n,k}$. To see this, write $\Psi(\sigma) = x_1^{b_1} \cdots x_n^{b_n}$. Since $\Psi(\sigma) \in \mathcal{M}_{n,k}$ we know that $0 \leq b_j \leq kr$ for all $1 \leq j \leq n$. If $\Psi(\sigma) \nmid x_1^{a_1} \cdots x_n^{a_n}$, choose $1 \leq j \leq n$ with $a_j < b_j$; by the last sentence we know $j \notin S$. A direct check shows that $\mathbf{x}(S \cup \{j\})^r \mid \Psi(\sigma)$, which contradicts $\Psi(\sigma) \in \mathcal{M}_{n,k}$. We conclude that $\Psi(\sigma) \in \mathcal{A}_{n,k}$. This completes the proof that $\Psi(\mathcal{F}_{n,k}) \subseteq \mathcal{A}_{n,k}$.

To prove the second assertion of the theorem, one verifies $\Psi(\mathcal{OP}_{n,k}) \subseteq \mathcal{B}_{n,k}$. The argument follows a similar inductive pattern and is left to the reader. \square

For example, consider the case $(n, k, r) = (5, 3, 2)$. The shuffles of $(1, 3, 5)$ and $(6, 6)$ are the ten sequences

$$\begin{array}{cccccc} (1, 3, 5, 6, 6) & (1, 3, 6, 5, 6) & (1, 6, 3, 5, 6) & (6, 1, 3, 5, 6) & (1, 3, 6, 6, 5) \\ (1, 6, 3, 6, 5) & (6, 1, 3, 6, 5) & (1, 6, 6, 3, 5) & (6, 1, 6, 3, 5) & (6, 6, 1, 3, 5), \end{array}$$

so that the standard monomial basis $\mathcal{M}_{5,3}$ of $R_{5,3}$ with respect to the lexicographic term order consists of those monomials $x_1^{a_1} \cdots x_5^{a_5}$ whose exponent sequence (a_1, \dots, a_5) is componentwise \leq at least one of these ten sequences. On the other hand, the shuffles of $(1, 3, 5)$ and $(5, 5)$ are the six sequences

$$(1, 3, 5, 5, 5) \quad (1, 5, 3, 5, 5) \quad (5, 1, 3, 5, 5) \quad (1, 5, 5, 3, 5) \quad (5, 1, 5, 3, 5) \quad (5, 5, 1, 3, 5),$$

so that the standard monomial basis $\mathcal{N}_{5,3}$ of $S_{5,3}$ consists of those monomials $x_1^{a_1} \cdots x_5^{a_5}$ where (a_1, \dots, a_5) is componentwise \leq at least one of these six sequences.

The next result gives the reduced Gröbner bases of the ideals $I_{n,k}$ and $J_{n,k}$. It is the extension of [14, Thm. 4.14] to $r \geq 2$.

Theorem 4.14. *Endow monomials in $\mathbb{C}[\mathbf{x}_n]$ with the lexicographic term order.*

- The variable powers $x_1^{kr+1}, \dots, x_n^{kr+1}$, together with the polynomials

$$\overline{\kappa_{\gamma(S)}(\mathbf{x}_n^r)}$$

for $S \subseteq [n]$ with $|S| = n - k + 1$, form a Gröbner basis for the ideal $I_{n,k} \subseteq \mathbb{C}[\mathbf{x}_n]$. If $n > k > 0$, this Gröbner basis is reduced.

- The variable powers $x_1^{kr}, \dots, x_n^{kr}$, together with the polynomials

$$\overline{\kappa_{\gamma(S)}(\mathbf{x}_n^r)}$$

for $S \subseteq [n-1]$ with $|S| = n - k + 1$, form a Gröbner basis for the ideal $J_{n,k} \subseteq \mathbb{C}[\mathbf{x}_n]$. If $n > k > 0$, this Gröbner basis is reduced.

Proof. By Lemma 3.4, the relevant polynomials $\overline{\kappa_{\gamma(S)}(\mathbf{x}_n^r)}$ lie in the ideals $I_{n,k}$ and $J_{n,k}$; the given variable powers are generators of these ideals. By Theorem 4.10, the number of monomials which do not divide any of the initial terms of the given polynomials equals the dimension of the corresponding quotient ring in either case. It follows that the given sets of polynomials are Gröbner bases for $I_{n,k}$ and $J_{n,k}$.

Suppose $n > k > 0$. By Lemma 3.3, for any distinct polynomials f, g listed in either bullet point, the leading monomial of f has coefficient 1 and does not divide any monomial in g . This implies the claim about reducedness. \square

5. GENERALIZED DESCENT MONOMIAL BASIS

5.1. A straightening algorithm. For an r -colored permutation $g = \pi_1^{c_1} \cdots \pi_n^{c_n} \in G_n$, let $d(g) = (d_1(g), \dots, d_n(g))$ be the sequence of nonnegative integers given by

$$(5.1) \quad d_i(g) := |\{j \in \text{Des}(\pi_1^{c_1} \cdots \pi_n^{c_n}) : j \geq i\}|.$$

We have $d_1(g) = \text{des}(g)$ and $d_1(g) \geq \cdots \geq d_n(g)$. Following Bango and Biagioli [4], we define the *descent monomial* $b_g \in \mathbb{C}[\mathbf{x}_n]$ by the equation

$$(5.2) \quad b_g := \prod_{i=1}^n x_{\pi_i}^{r d_i(g) + c_i}.$$

When $r = 1$, the monomials b_g were introduced by Garisa [8] and further studied by Garsia and Stanton [10]. Garsia [8] proved that the collection of monomials $\{b_g : g \in \mathfrak{S}_n\}$ descends to a basis for the coinvariant algebra attached to \mathfrak{S}_n . When $r = 2$, a slightly different family of monomials was introduced by Adin, Brenti, and Roichman [1]; they proved that their monomials descend to a basis for the coinvariant algebra attached to the hyperoctohedral group. Bango and

Biagioli [4] introduced the collection of monomials above; they proved that they descend to a basis for the coinvariant algebra attached to G_n (and, more generally, that an appropriate subset of them descend to a basis of the coinvariant algebra for the $G(r, p, n)$ family of complex reflection groups).

We will find it convenient to extend the definition of b_g somewhat to ‘partial colored permutations’ $g = \pi_1^{c_1} \dots \pi_m^{c_m}$, where π_1, \dots, π_m are distinct integers in $[n]$ and $0 \leq c_1, \dots, c_m \leq r-1$ are colors. The formulae (5.1) and (5.2) still make sense in this case and define a monomial $b_g \in \mathbb{C}[\mathbf{x}_n]$.

As an example of descent monomials, consider the case $(n, r) = (8, 3)$ and $g = \pi_1^{c_1} \dots \pi_8^{c_8} = 3^2 7^0 1^1 6^1 8^1 2^0 4^2 5^1 \in G_8$. We calculate $\text{Des}(g) = \{2, 6\}$, so that $d(g) = (2, 2, 1, 1, 1, 0, 0)$. The monomial $b_g \in \mathbb{C}[\mathbf{x}_8]$ is given by

$$b_g = x_3^8 x_7^6 x_1^4 x_6^4 x_8^3 x_2^3 x_4^2 x_5^1.$$

Let $\bar{g} = 6^1 8^1 2^0 4^2 5^1$ be the sequence obtained by erasing the first three letters of g . We leave it for the reader to check that

$$b_{\bar{g}} = x_6^4 x_8^4 x_2^3 x_4^2 x_5^1,$$

so that $b_{\bar{g}}$ is obtained by truncating b_g . We formalize this as an observation.

Observation 5.1. *Let $g = \pi_1^{c_1} \dots \pi_n^{c_n} \in G_n$ and let $\bar{g} = \pi_m^{c_m} \dots \pi_n^{c_n}$ for some $1 \leq m \leq n$. If $b_g = x_{\pi_1}^{a_1} \dots x_{\pi_n}^{a_n}$, then $b_{\bar{g}} = x_{\pi_m}^{a_m} \dots x_{\pi_n}^{a_n}$.*

The most important property of the b_g monomials will be a related *Straightening Lemma* of Bango and Biagioli [4] (see also [1]). This lemma uses a certain partial order on monomials. In order to define this partial order, we will attach colored permutations to monomials as follows.

Definition 5.2. Let $m = x_1^{a_1} \dots x_n^{a_n}$ be a monomial in $\mathbb{C}[\mathbf{x}_n]$. Let

$$g(m) = \pi_1^{c_1} \dots \pi_n^{c_n} \in G_n$$

be the r -colored permutation determined uniquely by the following conditions:

- $a_{\pi_i} \geq a_{\pi_{i+1}}$ for all $1 \leq i < n$,
- if $a_{\pi_i} = a_{\pi_{i+1}}$ then $\pi_i < \pi_{i+1}$, and
- $a_i \equiv c_i \pmod{r}$.

If $m = x_1^{a_1} \dots x_n^{a_n}$ is a monomial in $\mathbb{C}[\mathbf{x}_n]$, let $\lambda(m) = (\lambda(m)_1 \geq \dots \geq \lambda(m)_n)$ be the nonincreasing rearrangement of the exponent sequence (a_1, \dots, a_n) . The following partial order on monomials was introduced in [1, Sec. 3.3].

Definition 5.3. Let $m, m' \in \mathbb{C}[\mathbf{x}_n]$ be monomials and let $g(m) = \pi_1^{c_1} \dots \pi_n^{c_n}$ and $g(m') = \sigma_1^{e_1} \dots \sigma_n^{e_n}$ be the elements of G_n determined by Definition 5.2

We write $m_1 \prec m_2$ if $\deg(m) = \deg(m')$ and one of the following conditions holds:

- $\lambda(m) <_{\text{dom}} \lambda(m')$, or
- $\lambda(m) = \lambda(m')$ and $\text{inv}(\pi) > \text{inv}(\sigma)$.

Observe the numbers $\text{inv}(\pi)$ and $\text{inv}(\sigma)$ appearing in the second bullet refer to the inversion numbers of the *uncolored* permutations $\pi, \sigma \in \mathfrak{S}_n$.

In order to state the Straightening Lemma, we will need to attach a length n sequence $\mu(m) = (\mu(m)_1 \geq \dots \geq \mu(m)_n)$ of nonnegative integers to any monomial m . The basic tool for doing this is as follows; its proof is similar to that of [1, Claim 5.1].

Lemma 5.4. *Let $m = x_1^{a_1} \dots x_n^{a_n} \in \mathbb{C}[\mathbf{x}_n]$ be a monomial, let $g(m) = \pi_1^{c_1} \dots \pi_n^{c_n} \in G_n$ be the associated group element, and let $d(m) := d(g(m)) = (d_1 \geq \dots \geq d_n)$. The sequence*

$$(5.3) \quad a_{\pi_1} - r d_1 - c_1, \dots, a_{\pi_n} - r d_n - c_n$$

of exponents of $\frac{m}{b_{g(m)}}$ is a weakly decreasing sequence of nonnegative multiples of r .

Lemma 5.4 justifies the following definition.

Definition 5.5. Let $m = x_1^{a_1} \cdots x_n^{a_n}$ be a monomial and let $(a_{\pi_1} - rd_1 - c_1 \geq \cdots \geq a_{\pi_n} - rd_n - c_n)$ be the weakly decreasing sequence of nonnegative multiples of r guaranteed by Lemma 5.4. Let $\mu(m) = (\mu(m)_1, \dots, \mu(m)_n)$ be the partition *conjugate to* the partition

$$\left(\frac{a_{\pi_1} - rd_1 - c_1}{r}, \dots, \frac{a_{\pi_n} - rd_n - c_n}{r} \right).$$

As an example, consider $(n, r) = (8, 3)$ and $m = x_1^7 x_2^3 x_3^{14} x_4^2 x_5^1 x_6^7 x_7^{12} x_8^7$. We have $\lambda(m) = (14, 12, 7, 7, 3, 2, 1)$. We calculate $g(m) \in G_8$ to be $g(m) = 3^2 7^0 1^1 6^1 8^1 2^0 4^2 5^1$. From this it follows that $d(m) = (2, 2, 1, 1, 1, 1, 0, 0)$. The sequence $\mu(m)$ is determined by the equation

$$3 \cdot \mu(m)' = \lambda(m) - 3 \cdot d(m) - (2, 0, 1, 1, 1, 0, 2, 1),$$

from which it follows that $\mu(m)' = (2, 2, 1, 1, 1, 0, 0, 0)$ and $\mu(m) = (5, 2, 0, 0, 0, 0, 0, 0)$.

The Straightening Lemma of Bango and Biagioli [4] for monomials in $\mathbb{C}[\mathbf{x}_n]$ is as follows.

Lemma 5.6. (*Bango-Biagioli* [4]) *Let $m = x_1^{a_1} \cdots x_n^{a_n}$ be a monomial in $\mathbb{C}[\mathbf{x}_n]$. We have*

$$(5.4) \quad m = e_{\mu(m)}(\mathbf{x}_n^r) \cdot b_{g(m)} + \Sigma,$$

where Σ is a linear combination of monomials $m' \in \mathbb{C}[\mathbf{x}_n]$ which satisfy $m' \prec m$.

5.2. The rings $S_{n,k}$. We are ready to introduce our descent-type monomials for the rings $S_{n,k}$. This is an extension to $r \geq 1$ of the (n, k) -Garsia-Stanton monomials of [14, Sec. 5].

Definition 5.7. Let $n \geq k$. The collection $\mathcal{D}_{n,k}$ of (n, k) -descent monomials consists of all monomials in $\mathbb{C}[\mathbf{x}_n]$ of the form

$$(5.5) \quad b_g \cdot x_{\pi_1}^{r i_1} \cdots x_{\pi_{n-k}}^{r i_{n-k}},$$

where $g \in G_n$ satisfies $\text{des}(g) < k$ and the integer sequence (i_1, \dots, i_{n-k}) satisfies

$$k - \text{des}(g) > i_1 \geq \cdots \geq i_{n-k} \geq 0.$$

As an example, consider $(n, k, r) = (7, 5, 2)$ and let $g = 2^1 5^0 6^1 1^0 3^1 4^0 7^0 \in G_7$. It follows that $\text{Des}(g) = \{2, 4\}$ so that $\text{des}(g) = 2$ and $k - \text{des}(g) = 3$. We have

$$b_g = x_2^5 x_5^4 x_6^3 x_1^2 x_3^1,$$

so that Definition 5.7 gives rise to the following monomials in $\mathcal{D}_{7,5}$:

$$\begin{aligned} & x_2^5 x_5^4 x_6^3 x_1^2 x_3^1, \quad x_2^5 x_5^4 x_6^3 x_1^2 x_3^1 \cdot x_2^2, \quad x_2^5 x_5^4 x_6^3 x_1^2 x_3^1 \cdot x_2^4, \\ & x_2^5 x_5^4 x_6^3 x_1^2 x_3^1 \cdot x_2^2 x_5^2, \quad x_2^5 x_5^4 x_6^3 x_1^2 x_3^1 \cdot x_2^4 x_5^2, \quad x_2^5 x_5^4 x_6^3 x_1^2 x_3^1 \cdot x_2^4 x_5^4. \end{aligned}$$

By considering the possibilities for the sequence $(i_1 \geq \cdots \geq i_{n-k})$, we see that

$$(5.6) \quad |\mathcal{D}_{n,k}| \leq \sum_{g \in G_n} \binom{n - \text{des}(g) - 1}{n - k} = \sum_{g \in G_n} \binom{\text{asc}(g)}{n - k}$$

(where we have an inequality because *a priori* two monomials produced by Definition 5.7 for different choices of g could coincide). If we consider an ‘ascent-starred’ model for elements of $\mathcal{OP}_{n,k}$, e.g.

$$2_*^1 5_*^1 1^0 6^3 4_*^2 3^1 \in \mathcal{OP}_{6,3},$$

we see that

$$(5.7) \quad |\mathcal{D}_{n,k}| \leq |\mathcal{OP}_{n,k}| = \dim(S_{n,k}).$$

Our next theorem implies $|\mathcal{D}_{n,k}| = \dim(S_{n,k})$.

Theorem 5.8. *The collection $\mathcal{D}_{n,k}$ of (n, k) -descent monomials descends to a basis of the quotient ring $S_{n,k}$.*

Proof. By Equation 5.7, we need only show that $\mathcal{D}_{n,k}$ descends to a spanning set of the quotient ring $S_{n,k}$. To this end, let $m = x_1^{a_1} \cdots x_n^{a_n} \in \mathbb{C}[\mathbf{x}_n]$ be a monomial. We will show that the coset $m + J_{n,k}$ lies in the span of $\mathcal{D}_{n,k}$ by induction on the partial order \prec .

Suppose m is minimal with respect to the partial order \prec . Let us consider the exponent sequence (a_1, \dots, a_n) of m . By \prec -minimality, we have

$$(a_1, \dots, a_n) = (\underbrace{a, \dots, a}_p, \underbrace{a+1, \dots, a+1}_{n-p})$$

for some integers $a \geq 0$ and $0 < p \leq n$. Our analysis breaks into cases depending on the values of a and p .

- If $a \geq r$ then $e_n(\mathbf{x}_n^r) \mid m$, so that $m \equiv 0$ in the quotient $S_{n,k}$.
- If $0 \leq a < r$ and $p = n$, then $m = b_g$ where

$$g = 1^a 2^a \cdots n^a \in G_n.$$

- If $0 \leq a < r-1$ and $p < n$, then $m = b_g$ where

$$g = (p+1)^{a+1} (p+2)^{a+1} \cdots n^{a+1} 1^a 2^a \cdots p^a \in G_n.$$

- If $a = r-1$ and $0 < p < n$, then $m = b_g$ where

$$g = (p+1)^0 (p+2)^0 \cdots n^0 1^{r-1} 2^{r-1} \cdots p^{r-1} \in G_n.$$

We conclude that $m + J_{n,k}$ lies in the span of $\mathcal{D}_{n,k}$.

Now let $m = x_1^{a_1} \cdots x_n^{a_n}$ be an arbitrary monomial in $\mathbb{C}[\mathbf{x}_n]$. We inductively assume that for any monomial m' in $\mathbb{C}[\mathbf{x}_n]$ which satisfies $m' \prec m$, the coset $m' + J_{n,k}$ lies in the span of $\mathcal{D}_{n,k}$. We apply the Straightening Lemma 5.6 to m , which yields

$$m = e_{\mu(m)}(\mathbf{x}_n^r) \cdot b_{g(m)} + \Sigma,$$

where Σ is a linear combination of monomials $m' \prec m$; by induction, the ring element $\Sigma + J_{n,k}$ lies in the span of $\mathcal{D}_{n,k}$.

Write $d(m) = (d_1, \dots, d_n)$ and $g(m) = (\pi_1 \dots \pi_n, c_1 \dots c_n)$. Since $d_1 = \text{des}(g(m))$, if $\text{des}(g(m)) \geq k$, we would have $x_{\pi_1}^{kr} \mid b_{g(m)}$, so that $m \equiv \Sigma$ modulo $J_{n,k}$ and m lies in the span of $\mathcal{D}_{n,k}$. Similarly, if $\mu(m)_1 \geq n - k + 1$, then $e_{\mu(m)_1}(\mathbf{x}_n^r) \mid (e_{\mu(m)}(\mathbf{x}_n^r) \cdot b_{g(m)})$, so that again $m \equiv \Sigma$ modulo $J_{n,k}$ and m lies in the span of $\mathcal{D}_{n,k}$.

By the last paragraph, we may assume that

$$\text{des}(g(m)) < k \text{ and } \mu(m)_1 \leq n - k.$$

We have the identity

$$(5.8) \quad m = b_{g(m)} \cdot x_{\pi_1}^{r \cdot \mu(m)'_1} \cdots x_{\pi_n}^{r \cdot \mu(m)'_n},$$

where $\mu(m)'$ is the partition conjugate to $\mu(m)$. Since $\mu(m)_1 \leq n - k$, we may rewrite this identity as

$$(5.9) \quad m = b_{g(m)} \cdot x_{\pi_1}^{r \cdot \mu(m)'_1} \cdots x_{\pi_{n-k}}^{r \cdot \mu(m)'_{n-k}},$$

where the sequence $\mu(m)'_1, \dots, \mu(m)'_{n-k}$ is weakly decreasing. If $\mu(m)'_1 < k - \text{des}(g)$, we have $m \in \mathcal{D}_{n,k}$. If $\mu(m)'_1 \geq k - \text{des}(g)$, since $r \cdot \text{des}(g)$ is \leq the power of x_{π_1} in $b_{g(m)}$, we have $x_{\pi_1}^{kr} \mid m$, so that $m \equiv \Sigma$ modulo $J_{n,k}$. In either case, we have that $m + J_{n,k}$ lies in the span of $\mathcal{D}_{n,k}$. \square

5.3. The rings $R_{n,k}$. Our aim is to expand our set of monomials $\mathcal{D}_{n,k}$ to a larger set of monomials $\mathcal{ED}_{n,k}$ (the ‘extended’ descent monomials) which will descend to a basis for the rings $R_{n,k}$.

Definition 5.9. Let the *extended (n,k) -descent monomials* $\mathcal{ED}_{n,k}$ be the set of monomials of the form

$$(5.10) \quad \left(\prod_{j=1}^z x_{\pi_j}^{kr} \right) \cdot b_{\pi_{z+1}^{c_{z+1}} \dots \pi_n^{c_n}} \cdot \left(x_{\pi_{z+1}}^{r \cdot i_{z+1}} x_{\pi_{z+2}}^{r \cdot i_{z+2}} \dots x_{\pi_{z+n-k}}^{r \cdot i_{z+n-k}} \right),$$

where

- we have $0 \leq z \leq n - k$,
- $\pi_1^{c_1} \dots \pi_n^{c_n} \in G_n$ is a colored permutation whose length $n - z$ suffix $\pi_{z+1}^{c_{z+1}} \dots \pi_n^{c_n}$ satisfies $\text{des}(\pi_{z+1}^{c_{z+1}} \dots \pi_n^{c_n}) < k$, and
- we have

$$k - \text{des}(\pi_{z+1}^{c_{z+1}} \dots \pi_n^{c_n}) > i_{z+1} \geq i_{z+2} \geq \dots \geq i_{z+n-k} \geq 0.$$

We also set $\mathcal{ED}_{n,0} := \{1\}$.

As an example of Definition 5.9, let $(n, k, r) = (7, 3, 2)$, let $z = 2$, and consider the group element $5^1 1^1 2^0 6^0 7^0 4^1 3^0 \in G_7$. We have $\text{des}(2^0 6^0 7^0 4^1 3^0) = 1$, so that $k - \text{des}(2^0 6^0 7^0 4^1 3^0) = 2$. Moreover, we have

$$b_{2^0 6^0 7^0 4^1 3^0} = x_2^2 x_6^2 x_7^2 x_4^1,$$

so that we get the following monomials in $\mathcal{ED}_{7,3}$:

$$(x_5^6 x_1^6) \cdot (x_2^2 x_6^2 x_7^2 x_4^1), \quad (x_5^6 x_1^6) \cdot (x_2^2 x_6^2 x_7^2 x_4^1) \cdot (x_2^2), \quad (x_5^6 x_1^6) \cdot (x_2^2 x_6^2 x_7^2 x_4^1) \cdot (x_2^2 x_6^2).$$

Observe that the monomial defined in (5.10) depends only on the set of letters $\{\pi_1, \dots, \pi_z\}$ contained in the length z prefix $\pi_1^{c_1} \dots \pi_z^{c_z}$ of $\pi_1^{c_1} \dots \pi_n^{c_n}$. We can therefore form a typical monomial in $\mathcal{ED}_{n,k}$ by choosing $0 \leq z \leq n - k$, then choosing a set $Z \subseteq [n]$ with $|Z| = z$, then forming a typical element of $\mathcal{D}_{n-z,k}$ on the variable set $\{x_j : j \in [n] - Z\}$, and finally multiplying by the product $\prod_{j \in Z} x_j^{kr}$. By Theorem 5.8, there are $|\mathcal{OP}_{n-z,k}|$ monomials in $\mathcal{D}_{n-z,k}$, and all of the exponents in these monomials are $< kr$. It follows that

$$(5.11) \quad |\mathcal{ED}_{n,k}| = \sum_{z=0}^{n-k} \binom{n}{z} |\mathcal{D}_{n-z,k}| = \sum_{z=0}^{n-k} \binom{n}{z} |\mathcal{OP}_{n-z,k}| = |\mathcal{F}_{n,k}| = \dim(R_{n,k}).$$

We will show $\mathcal{ED}_{n,k}$ descends to a spanning set of $R_{n,k}$, and hence descends to a basis of $R_{n,k}$.

Theorem 5.10. *The set $\mathcal{ED}_{n,k}$ of extended (n,k) -descent monomials descends to a basis of $R_{n,k}$.*

Proof. Let $m = x_1^{a_1} \dots x_n^{a_n}$ be a monomial in $\mathbb{C}[\mathbf{x}_n]$. We argue that the coset $m + I_{n,k} \in R_{n,k}$ lies in the span of $\mathcal{ED}_{n,k}$.

Suppose first that m is minimal with respect to \prec . The exponent sequence (a_1, \dots, a_n) has the form

$$(a_1, \dots, a_n) = (\underbrace{a, \dots, a}_p, \underbrace{a+1, \dots, a+1}_{n-p})$$

for some $a \geq 0$ and $0 < p \leq n$. The same analysis as in the proof of Theorem 5.8 implies that $m \equiv 0 \pmod{I_{n,k}}$ or $m \in \mathcal{D}_{n,k} \subseteq \mathcal{ED}_{n,k}$.

Now let $m = x_1^{a_1} \dots x_n^{a_n} \in \mathbb{C}[\mathbf{x}_n]$ be an arbitrary monomial and form the sequence $d(m) = (d_1, \dots, d_n)$ and the colored permutation $g(m) = \pi_1^{c_1} \dots \pi_n^{c_n}$. Apply the Straightening Lemma 5.6 to write

$$(5.12) \quad m = e_{\mu(m)}(\mathbf{x}_n^r) \cdot b_{g(m)} + \Sigma,$$

where Σ is a linear combination of monomials $m' \in \mathbb{C}[\mathbf{x}_n]$ with $m' \prec m$.

We inductively assume that the ring element $\Sigma + I_{n,k}$ lies in the span of $\mathcal{ED}_{n,k}$. If $\mu(m)_1 \geq n-k+1$, then $m \equiv \Sigma \pmod{I_{n,k}}$, so that $m + I_{n,k}$ lies in the span of $\mathcal{ED}_{n,k}$. If $\text{des}(g(m)) > k+1$, then $x_{\pi_1}^{(k+1)r} \mid b_{g(m)}$, so that again $m \equiv \Sigma \pmod{I_{n,k}}$ and $m + I_{n,k}$ lies in the span of $\mathcal{ED}_{n,k}$.

By the last paragraph, we may assume

$$\mu(m)_1 \leq n-k \text{ and } \text{des}(g(m)) \leq k.$$

Our analysis breaks up into two cases depending on whether $\text{des}(g(m)) < k$ or $\text{des}(g(m)) = k$.

Case 1: $\mu(m)_1 \leq n-k$ and $\text{des}(g(m)) < k$.

If any element in the exponent sequence (a_1, \dots, a_n) of m is $> kr$, then $m \equiv 0 \pmod{I_{n,k}}$. We may therefore assume $a_j \leq kr$ for all j .

Since we have $\mu(m)_1 \leq n-k$, we have the identity

$$(5.13) \quad m = b_{g(m)} \cdot x_{\pi_1}^{r \cdot \mu(m)'_1} \cdots x_{\pi_{n-k}}^{r \cdot \mu(m)'_{n-k}}.$$

If $\mu(m)'_1 < k - \text{des}(g(m))$, we have $m \in \mathcal{D}_{n,k} \subseteq \mathcal{ED}_{n,k}$. If $\mu(m)'_1 > k - \text{des}(g(m))$, we have $x_{\pi_1}^{(k+1)r} \mid m$, which contradicts $a_{\pi_1} \leq kr$.

By the last paragraph, we may assume $\mu(m)'_1 = k - \text{des}(g(m))$. Since every term in the weakly decreasing sequence $(a_{\pi_1}, \dots, a_{\pi_n})$ is $\leq kr$, there exists an index $1 \leq z \leq n$ such that $(a_{\pi_1}, \dots, a_{\pi_n}) = (kr, \dots, kr, a_{\pi_{z+1}}, \dots, a_{\pi_n})$, where $a_{\pi_{z+1}} < kr$. Since every exponent in $b_{g(m)}$ is $< kr$, we in fact have $1 \leq z \leq n-k$.

Let \bar{g} be the partial colored permutation $\bar{g} := \pi_{z+1}^{c_{z+1}} \cdots \pi_n^{c_n}$. Applying Observation 5.1, we have

$$(5.14) \quad m = b_{g(m)} \cdot x_{\pi_1}^{r \cdot \mu(m)'_1} \cdots x_{\pi_{n-k}}^{r \cdot \mu(m)'_{n-k}}$$

$$(5.15) \quad = \left(\prod_{j=1}^z x_{\pi_j}^{kr} \right) \cdot b_{\bar{g}} \cdot x_{\pi_{z+1}}^{r \cdot \mu(m)'_{z+1}} \cdots x_{\pi_{n-k}}^{r \cdot \mu(m)'_{n-k}},$$

for $1 \leq z \leq n-k$. The monomial $b_{\bar{g}} \cdot x_{\pi_{z+1}}^{r \cdot \mu(m)'_{z+1}} \cdots x_{\pi_{n-k}}^{r \cdot \mu(m)'_{n-k}}$ only involves the variables $x_{\pi_{z+1}}, \dots, x_{\pi_n}$, and every exponent in this product is $< kr$. If $\mu(m)'_{z+1} \geq k - \text{des}(\bar{g})$, we would have the divisibility $x_{\pi_{z+1}}^{kr} \mid b_{\bar{g}} \cdot x_{\pi_{z+1}}^{r \cdot \mu(m)'_{z+1}} \cdots x_{\pi_{n-k}}^{r \cdot \mu(m)'_{n-k}}$, which is a contradiction. It follows that $\mu(m)'_{z+1} < k - \text{des}(\bar{g})$, which implies that $m \in \mathcal{ED}_{n,k}$.

We conclude that the coset $m + I_{n,k}$ lies in the span of $\mathcal{ED}_{n,k}$, which completes this case.

Case 2: $\mu(m)_1 \leq n-k$ and $\text{des}(g(m)) = k$.

As in the previous case, we may assume that every exponent appearing in the monomial m is $\leq kr$. We again write

$$(5.16) \quad m = b_{g(m)} \cdot x_{\pi_1}^{r \cdot \mu(m)'_1} \cdots x_{\pi_{n-k}}^{r \cdot \mu(m)'_{n-k}}$$

and have $(a_{\pi_1} \geq \dots \geq a_{\pi_n}) = (kr, \dots, kr, a_{\pi_{z+1}}, \dots, a_{\pi_n})$ for some $1 \leq z \leq n-k$. Define the partial colored permutation $\bar{g} := \pi_{z+1}^{c_{z+1}} \cdots \pi_n^{c_n}$.

Since the exponent of $x_{\pi_{z+1}}$ in m is $\geq r \cdot \text{des}(\bar{g})$, we have $\text{des}(\bar{g}) < k$. If $\mu(m)'_{z+1} \geq k - \text{des}(\bar{g})$, the exponent of $x_{\pi_{z+1}}$ in m would be $\geq kr$, so we must have $\mu(m)'_{z+1} < k - \text{des}(\bar{g})$. Using Observation 5.1 to write

$$(5.17) \quad m = b_{g(m)} \cdot x_{\pi_1}^{r \cdot \mu(m)'_1} \cdots x_{\pi_{n-k}}^{r \cdot \mu(m)'_{n-k}}$$

$$(5.18) \quad = \left(\prod_{j=1}^z x_{\pi_j}^{kr} \right) \cdot b_{\bar{g}} \cdot x_{\pi_{z+1}}^{r \cdot \mu(m)'_{z+1}} \cdots x_{\pi_{n-k}}^{r \cdot \mu(m)'_{n-k}},$$

we see that $m \in \mathcal{ED}_{n,k}$. □

The following lemma involving expansions of monomials m into the $\mathcal{ED}_{n,k}$ basis of $R_{n,k}$ will be useful in the next section. For $0 \leq z \leq n - k$, let $\mathcal{ED}_{n,k}(z)$ be the subset of monomials in $\mathcal{ED}_{n,k}$ which contain exactly z variables with power kr . We get a stratification

$$(5.19) \quad \mathcal{ED}_{n,k} = \mathcal{ED}_{n,k}(0) \uplus \mathcal{ED}_{n,k}(1) \uplus \cdots \uplus \mathcal{ED}_{n,k}(n - k).$$

For convenience, we set $\mathcal{ED}_{n,k}(z) = \emptyset$ for $z > n - k$.

Lemma 5.11. *Let (a_1, \dots, a_n) satisfy $0 \leq a_i \leq kr$ for all i , let $m = x_1^{a_1} \cdots x_n^{a_n} \in \mathbb{C}[\mathbf{x}_n]$ be the corresponding monomial, and let $z := |\{1 \leq i \leq n : a_i = kr\}|$. The expansion of $m + I_{n,k}$ in the basis $\mathcal{ED}_{n,k}$ of $R_{n,k}$ only involves terms in $\mathcal{ED}_{n,k}(0) \uplus \mathcal{ED}_{n,k}(1) \uplus \cdots \uplus \mathcal{ED}_{n,k}(z)$.*

Proof. Applying the Straightening Lemma 5.6 to m , we get

$$(5.20) \quad m = e_{\mu(m)}(\mathbf{x}_n^r) \cdot b_{g(m)} + \Sigma,$$

where Σ is a linear combination of monomials m' in $\mathbb{C}[\mathbf{x}_n]$ which satisfy $m' \prec m$. The proof of Theorem 5.10 shows that either

- the monomial m is an element of $\mathcal{ED}_{n,k}$, and hence an element of $\mathcal{ED}_{n,k}(z)$, or
- we have $m \equiv \Sigma \pmod{I_{n,k}}$.

If the first bullet holds, we are done. We may therefore assume that $m \equiv \Sigma \pmod{I_{n,k}}$.

Let $m' = x_1^{a'_1} \cdots x_n^{a'_n}$ be a monomial appearing in Σ . The dominance relation $\lambda(m') \leq_{\text{dom}} \lambda(m)$ implies $|\{1 \leq i \leq n : a'_i = kr\}| \leq z$. We may therefore apply the logic of the last paragraph to each such monomial m' , and iterate. \square

6. FROBENIUS SERIES

In this section we will determine the graded isomorphism types of the rings $R_{n,k}$ and $S_{n,k}$. When $r = 1$, this was carried out for the rings $S_{n,k}$ in [14, Sec. 6]. It turns out that the methods developed in [14, Sec. 6] generalize fairly readily to the S rings, but not the R rings. Our approach will be to describe the R rings in terms of the S rings, and then describe the isomorphism type of the S rings.

6.1. Relating R and S . In this section, we describe the graded isomorphism type of $R_{n,k}$ in terms of the rings $S_{n,k}$. The result here is as follows.

Proposition 6.1. *We have an isomorphism of graded G_n -modules*

$$(6.1) \quad R_{n,k} \cong \bigoplus_{z=0}^{n-k} \text{Ind}_{G_{(n-z,z)}}^{G_n} (S_{n-z,k}^r \otimes \mathbb{C}_{krz}).$$

Here \mathbb{C}_{krz} is a copy of the trivial 1-dimensional representation of G_z sitting in degree krz .

Equivalently, we have the identity

$$(6.2) \quad \text{grFrob}(R_{n,k}; q) = \sum_{z=0}^{n-k} q^{krz} \mathbf{s}_{(\emptyset, \dots, \emptyset, (z))}(\mathbf{x}) \cdot \text{grFrob}(S_{n-z,k}^r; q).$$

Proof. For $0 \leq z \leq n - k$, let $R_{n,k}(z)$ be the subspace of $R_{n,k}$ given by

$$(6.3) \quad R_{n,k}(z) := \text{span}_{\mathbb{C}}\{x_1^{a_1} \cdots x_n^{a_n} + I_{n,k} : 0 \leq a_i \leq kr \text{ and at most } z \text{ of } a_1, \dots, a_n \text{ equal } kr\}.$$

It is clear that $R_{n,k}(z)$ is graded and stable under the action of G_n . We also have a filtration

$$(6.4) \quad R_{n,k}(0) \subseteq R_{n,k}(1) \subseteq \cdots \subseteq R_{n,k}(n - k) = R_{n,k}.$$

It follows that there is an isomorphism of graded G_n -modules

$$(6.5) \quad R_{n,k} \cong Q_{n,k}^r(0) \oplus Q_{n,k}^r(1) \oplus \cdots \oplus Q_{n,k}^r(n - k),$$

where $Q_{n,k}^r(z) := R_{n,k}(z)/R_{n,k}(z - 1)$.

Consider the stratification $\mathcal{ED}_{n,k} = \mathcal{ED}_{n,k}(0) \uplus \mathcal{ED}_{n,k}(1) \uplus \cdots \uplus \mathcal{ED}_{n,k}(n-k)$ of the basis $\mathcal{ED}_{n,k}$ of $R_{n,k}$. The containment $\mathcal{ED}_{n,k}(z') \subseteq R_{n,k}(z)$ for $z' \leq z$ implies

$$(6.6) \quad \dim(R_{n,k}(z)) \geq |\mathcal{ED}_{n,k}(0)| + |\mathcal{ED}_{n,k}(1)| + \cdots + |\mathcal{ED}_{n,k}(z)|.$$

On the other hand, Lemma 5.11 implies that $R_{n,k}(z)$ is spanned by (the image of the monomials in) $\biguplus_{z'=0}^z \mathcal{ED}_{n,k}(z')$. It follows that

$$(6.7) \quad \dim(R_{n,k}(z)) = |\mathcal{ED}_{n,k}(0)| + |\mathcal{ED}_{n,k}(1)| + \cdots + |\mathcal{ED}_{n,k}(z)|.$$

and $\biguplus_{z'=0}^z \mathcal{ED}_{n,k}(z')$ descends to a basis of $R_{n,k}(z)$. Consequently, the set $\mathcal{ED}_{n,k}(z)$ descends to a basis for $Q_{n,k}^r(z)$.

Fix $0 \leq z \leq n-k$. It follows from the definition of $\mathcal{ED}_{n,k}(z)$ that

$$(6.8) \quad \dim(Q_{n,k}^r(z)) = |\mathcal{ED}_{n,k}(z)| = \binom{n}{z} \cdot |\mathcal{OP}_{n-z,k}| = \binom{n}{z} \cdot \dim(S_{n,k}),$$

which coincides with the dimension of $\text{Ind}_{G_{(n-z,z)}}^{G_n}(S_{n-z,k}^r \otimes \mathbb{C}_{krz})$. We claim that we have an isomorphism of graded G_n -modules

$$(6.9) \quad Q_{n,k}^r(z) \cong \text{Ind}_{G_{(n-z,z)}}^{G_n}(S_{n-z,k}^r \otimes \mathbb{C}_{krz}).$$

In order to prove the isomorphism (6.9), for any $T \subseteq [n]$, let $G_{[n]-T}$ be the group of r -colored permutations on the index set $[n] - T$ and let $S_{n-z,k}(T)$ be the module $S_{n-z,k}$ in the variable set $\{x_j : j \in T\}$. Any group element $g \in G_{[n]-T}$ acts trivially on the product $\prod_{j \notin T} x_j^{kr}$. We may therefore interpret the induction on the right-hand side of (6.9) as

$$(6.10) \quad \text{Ind}_{G_{(z,n-z)}}^{G_n}(S_{n-z,k} \otimes \mathbb{C}_{krz}) \cong \bigoplus_{|T|=n-z} S_{n-z,k}(T) \otimes \text{span} \left\{ \prod_{j \notin T} x_j^{kr} \right\},$$

which reduces our task to proving

$$(6.11) \quad Q_{n,k}^r(z) \cong \bigoplus_{|T|=n-z} S_{n-z,k}(T) \otimes \text{span} \left\{ \prod_{j \notin T} x_j^{kr} \right\}.$$

The set of monomials $\mathcal{ES}_{n,k}(z)$ in $\mathbb{C}[\mathbf{x}_n]$ descends to a vector space basis of the graded modules appearing on either side of (6.11); the corresponding identification of cosets gives rise to an isomorphism

$$(6.12) \quad \varphi : Q_{n,k}^r(z) \rightarrow \bigoplus_{|T|=n-z} S_{n-z,k}^r(T) \otimes \text{span} \left\{ \prod_{j \notin T} x_j^{kr} \right\}.$$

of graded vector spaces. It is clear that φ commutes with the action of the diagonal subgroup $\mathbb{Z}_r \times \cdots \times \mathbb{Z}_r \subseteq G_n$; we need only show that φ commutes with the action of \mathfrak{S}_n .

The proof that the map φ commutes with the action of \mathfrak{S}_n uses straightening. Let $m = x_1^{a_1} \cdots x_n^{a_n} \in \mathcal{ED}_{n,k}(z)$ be a typical basis element and let $\pi.m = x_{\pi_1}^{a_1} \cdots x_{\pi_n}^{a_n}$ be the image of m under a typical permutation $\pi \in \mathfrak{S}_n$.

If $\pi.m \in \mathcal{ED}_{n,k}(z)$ the definition of φ yields $\varphi(\pi.m) = \pi.\varphi(m)$.

If $\pi.m \notin \mathcal{ED}_{n,k}(z)$, by Lemma 5.6 we can write $\pi.m = e_{\mu(\pi.m)}(\mathbf{x}_n^r) \cdot b_{g(\pi.m)} + \Sigma$, where Σ is a linear combination of monomials in $\mathbb{C}[\mathbf{x}_n]$ which are $\prec \pi.m$. As in the proof of Lemma 5.11, since $m \in \mathcal{ED}_{n,k}(z)$ but $\pi.m \notin \mathcal{ED}_{n,k}(z)$, we know that $\pi.m \equiv \Sigma$ in the modules on either side of Equation 6.11. Iterating this procedure, we see that $\pi.m$ has the same expansion into the bases induced from $\mathcal{ED}_{n,k}(z)$ on either side of Equation 6.11. This proves that the map φ is \mathfrak{S}_n -equivariant, so that φ is an isomorphism of graded G_n -modules. \square

6.2. The rings $S_{n,k,s}$. By Proposition 6.1, the graded isomorphism type of $R_{n,k}$ is determined by the graded isomorphism type of $S_{n,k}$. The remainder of this section will focus on the rings $S_{n,k}$. As in [14, Sec. 6], to determine the graded isomorphism type of $S_{n,k}$ we will introduce a more general class of quotients.

Definition 6.2. Let n, k, s be positive integers with $n \geq k \geq s$. Define $J_{n,k,s} \subseteq \mathbb{C}[\mathbf{x}_n]$ to be the ideal

$$J_{n,k,s} := \langle x_1^{kr}, \dots, x_n^{kr}, e_n(\mathbf{x}_n^r), e_{n-1}(\mathbf{x}_n^r), \dots, e_{n-s+1}(\mathbf{x}_n^r) \rangle.$$

Let $S_{n,k,s} := \mathbb{C}[\mathbf{x}_n]/J_{n,k,s}$ be the corresponding quotient ring.

When $s = k$ we have $J_{n,k,k} = J_{n,k}$, so that $S_{n,k,k} = S_{n,k}$. Our aim for the remainder of this section is to build a combinatorial model for the quotient $S_{n,k,s}$ using the point orbit technique of Section 4. To this end, for $n \geq k \geq s$ let $\mathcal{OP}_{n,k,s}$ denote the collection of r -colored k -block ordered set partitions $\sigma = (B_1 \mid \dots \mid B_k)$ of $[n + (k - s)]$ such that, for $1 \leq i \leq k - s$, we have $n + i \in B_{s+i}$ and $n + i$ has color 0. For example, we have

$$(2^0 3^2 \mid 1^2 6^0 \mid \mathbf{7}^0 \mid 5^1 7^2 8^0 \mid 4^1 9^0) \in \mathcal{OP}_{6,5,2}^3.$$

Given $\sigma \in \mathcal{OP}_{n,k,s}$, we will refer to the letters $n + 1, n + 2, \dots, n + (k - s)$ as *big*; the remaining letters will be called *small*. The group G_n acts on $\mathcal{OP}_{n,k,s}$ by acting on the small letters. We model this action with a point set as follows.

Definition 6.3. Fix positive real numbers $0 < \alpha_1 < \dots < \alpha_k$. Let $Z_{n,k,s} \subseteq \mathbb{C}^{n+(k-s)}$ be the collection of points $(z_1, \dots, z_n, z_{n+1}, \dots, z_{n+k-s})$ such that

- we have $z_i \in \{\zeta^c \alpha_j : 0 \leq c \leq r - 1, 1 \leq j \leq k\}$ for all $1 \leq i \leq n + (k - s)$,
- we have $\{\alpha_1, \dots, \alpha_k\} = \{|z_1|, \dots, |z_n|\}$, and
- we have $z_{n+i} = \alpha_{s+i}$ for all $1 \leq i \leq k - s$.

It is evident that the point set $Z_{n,k,s}$ is stable under the action of G_n on the first n coordinates of $\mathbb{C}^{n+(k-s)}$ and that $Z_{n,k,s}$ is isomorphic to the action of G_n on $\mathcal{OP}_{n,k,s}$.

Let $\mathbf{I}(Z_{n,k,s}) \subseteq \mathbb{C}[\mathbf{x}_{n+(k-s)}]$ be the ideal of polynomials which vanish on $Y_{n,k,s}$ and let $\mathbf{T}(Y_{n,k,s}) \subseteq \mathbb{C}[\mathbf{x}_{n+(k-s)}]$ be the corresponding top component ideal. Since $x_{n+i} - \alpha_{n+i} \in \mathbf{I}(Y_{n,k,s})$ for all $1 \leq i \leq k - s$, we have $x_{n+i} \in \mathbf{T}(Y_{n,k,s})$. Let $\varepsilon : \mathbb{C}[\mathbf{x}_{n+(k-s)}] \rightarrow \mathbb{C}[\mathbf{x}_n]$ be the map which evaluates $x_{n+i} = 0$ for all $1 \leq i \leq k - s$ and let $T_{n,k,s} := \varepsilon(\mathbf{T}(Y_{n,k,s}))$ be the image of $\mathbf{T}(Y_{n,k,s})$ under ε . Then $T_{n,k,s}$ is an ideal in $\mathbb{C}[\mathbf{x}_n]$ and we have an identification of G_n -modules

$$\mathbb{C}[\mathcal{OP}_{n,k,s}] \cong \mathbb{C}[\mathbf{x}_{n+(k-s)})/\mathbf{I}(Y_{n,k,s}) \cong \mathbb{C}[\mathbf{x}_{n+(k-s)})/\mathbf{T}(Y_{n,k,s}) \cong \mathbb{C}[\mathbf{x}_n]/T_{n,k,s}.$$

It will develop that $J_{n,k,s} = T_{n,k,s}$. We can generalize Lemma 4.2 to prove one containment right away.

Lemma 6.4. *We have $J_{n,k,s} \subseteq T_{n,k,s}$.*

Proof. We show that every generator of $J_{n,k,s}$ is contained in $T_{n,k,s}$.

For $1 \leq i \leq n$ we have $\prod_{j=1}^r \prod_{c=0}^{r-1} (x_i - \zeta^c \alpha_i) \in \mathbf{I}(Y_{n,k,s})$, so that $x_i^{kr} \in T_{n,k,s}$.

The proof of Lemma 4.2 shows that $e_j(\mathbf{x}_{n+(k-s)}^r) \in \mathbf{T}(Y_{n,k,s})$ for all $j \geq n - s + 1$. Applying the evaluation map ε gives $\varepsilon : e_j(\mathbf{x}_{n+(k-s)}^r) \mapsto e_j(\mathbf{x}_n^r) \in T_{n,k,s}$. \square

Proving the equality $J_{n,k,s} = T_{n,k,s}$ will involve a dimension count. To facilitate this, let us identify some terms in the initial ideal of $J_{n,k,s}$. The following is a generalization of Lemma 6.5; its proof is left to the reader.

Lemma 6.5. *Let $<$ be the lexicographic term order on monomials in $\mathbb{C}[\mathbf{x}_n]$ and let $\text{in}_{<}(J_{n,k,s})$ be the initial ideal of $J_{n,k,s}$. We have*

- $x_i^{kr} \in \text{in}_{<}(J_{n,k,s})$ for $1 \leq i \leq n$, and

- $\mathbf{x}(S)^r \in \text{in}_<(J_{n,k,s})$ for all $S \subseteq [n]$ with $|S| = n - s + 1$.

Lemma 6.5 motivates the following generalization of strongly (n, k) -nonskip monomials.

Definition 6.6. Let $\mathcal{N}_{n,k,s}$ be the collection of monomials $m \in \mathbb{C}[\mathbf{x}_n]$ such that

- $x_i^{kr} \nmid m$ for all $1 \leq i \leq m$, and
- $\mathbf{x}(S)^r \nmid m$ for all $S \subseteq [n]$ with $|S| = n - s + 1$.

By Lemma 6.5, the set $\mathcal{N}_{n,k,s}$ contains the standard monomial basis of $S_{n,k,s}$; we will prove that these two sets of monomials coincide. Let us first observe a relationship between the monomials in $\mathcal{N}_{n,k,s}$ and those in $\mathcal{N}_{n+(k-s),k}$.

Lemma 6.7. If $x_1^{a_1} \cdots x_n^{a_n} x_{n+1}^{a_{n+1}} \cdots x_{n+(k-s)}^{a_{n+(k-s)}} \in \mathcal{N}_{n+(k-s),k}$, then $x_1^{a_1} \cdots x_n^{a_n} \in \mathcal{N}_{n,k,s}$.

Conversely, if $x_1^{a_1} \cdots x_n^{a_n} \in \mathcal{N}_{n,k,s}$ and $0 \leq a_{n+1} < a_{n+2} < \cdots < a_{n+(k-s)} < kr$ satisfy

$$a_{n+1} \equiv a_{n+2} \equiv \cdots \equiv a_{n+(k-s)} \equiv i \pmod{r}$$

for some $0 \leq i \leq r-1$, then $x_1^{a_1} \cdots x_n^{a_n} x_{n+1}^{a_{n+1}} \cdots x_{n+(k-s)}^{a_{n+(k-s)}} \in \mathcal{N}_{n+(k-s),k}$.

Proof. The first statement is clear from the definitions of $\mathcal{N}_{n+(k-s),k}$ and $\mathcal{N}_{n,k,s}$. For the second statement, let $m' := x_1^{a_1} \cdots x_n^{a_n} \in \mathcal{N}_{n,k,s}$ and let $0 \leq a_{n+1} < a_{n+2} < \cdots < a_{n+(k-s)} < kr$ be as in the statement of the lemma. We argue that $m := x_1^{a_1} \cdots x_n^{a_n} x_{n+1}^{a_{n+1}} \cdots x_{n+(k-s)}^{a_{n+(k-s)}} \in \mathcal{N}_{n+(k-s),k}$.

Since $m' \in \mathcal{N}_{n,k,s}$, we know that $x_i^{kr} \nmid m'$ for $1 \leq i \leq n + (k-s)$. Let $S \subseteq [n + (k-s)]$ satisfy $|S| = n + (k-s)$. We need to show $\mathbf{x}(S)^r \nmid m$. If $S \subseteq [n]$, then $\mathbf{x}(S)^r \nmid m$ because $\mathbf{x}(S)^r \nmid m'$. On the other hand, if $n+i \in S$ for some $1 \leq i \leq k-s$, the power p_{n+i} of x_{n+i} in $\mathbf{x}(S)^r$ is $\geq r \cdot (s+i)$. However, our assumptions on $(a_{n+1}, a_{n+2}, \dots, a_{n+(k-s)})$ force $a_{n+i} < r \cdot (k - (s-i)) \leq r \cdot (s+i)$, which implies $\mathbf{x}(S)^r \nmid m$. \square

We use the map Ψ from Section 4 to count $\mathcal{N}_{n,k,s}$.

Lemma 6.8. We have $|\mathcal{N}_{n,k,s}| = |\mathcal{OP}_{n,k,s}|$.

Proof. Consider the bijection $\Psi : \mathcal{OP}_{n+(k-s),k} \rightarrow \mathcal{N}_{n+(k-s),k}$ from Section 4. We have $\mathcal{OP}_{n,k,s} \subseteq \mathcal{OP}_{n+(k-s),k}$. We leave it for the reader to check that

$$\Psi(\mathcal{OP}_{n,k,s}) = \mathcal{N}'_{n,k,s},$$

where $\mathcal{N}'_{n,k,s}$ consists of those monomials $x_1^{a_1} \cdots x_n^{a_n} x_{n+1}^{a_{n+1}} \cdots x_{n+(k-s)}^{a_{n+(k-s)}} \in \mathcal{N}_{n+(k-s),k}$ which satisfy

$$(a_{n+1}, a_{n+2}, \dots, a_{n+(k-s)}) = (rs + (r-1), r(s+1) + (r-1), \dots, r(k-1) + (r-1)).$$

(The $+(r-1)$ terms come from the fact that the letters $n+1, \dots, n+(k-s)$ all have color 0 and Ψ involves a *complementary* color contribution.) Lemma 6.7 applies to show $|\mathcal{N}'_{n,k,s}| = |\mathcal{N}_{n,k,s}|$. \square

We are ready to determine the ungraded isomorphism type of the G_n -module $S_{n,k,s}$.

Lemma 6.9. We have $S_{n,k,s} \cong \mathbb{C}[\mathcal{OP}_{n,k,s}]$. In particular, we have $\dim(S_{n,k,s}) = |\mathcal{OP}_{n,k,s}|$.

Proof. By Lemma 6.4 we have $\dim(S_{n,k,s}) \geq |\mathcal{OP}_{n,k,s}|$. Lemma 6.5 and Lemma 6.8 imply that the standard monomial basis of $S_{n,k,s}$ with respect to the lexicographic term order has size $\leq |\mathcal{N}_{n,k,s}| = |\mathcal{OP}_{n,k,s}|$, so that $\dim(S_{n,k,s}) = |\mathcal{OP}_{n,k,s}|$. Lemma 6.4 gives a G_n -module surjection $S_{n,k,s} \twoheadrightarrow \mathbb{C}[\mathcal{OP}_{n,k,s}]$; dimension counting shows that this surjection is an isomorphism. \square

6.3. Idempotents and $e_j(\mathbf{x}^{(i*)})^\perp$. For $1 \leq j \leq n$ and $1 \leq i \leq r$, we want to develop a module-theoretic analog of acting by the operator $e_j(\mathbf{x}^{(i*)})^\perp$ on Frobenius images. If V is a G_n -module, acting by $e_j(\mathbf{x}^{(i*)})^\perp$ on $\text{Frob}(V)$ will correspond to taking the image of V under a certain group algebra idempotent $\epsilon_{i,j} \in \mathbb{C}[G_n]$.

Let $1 \leq j \leq n$ and consider the corresponding parabolic subgroup $G_{(n-j,j)} = G_{n-j} \times G_j$ of G_n . The factor G_j acts on the *last* j letters $n-j+1, \dots, n-1, n$ of $\{1, 2, \dots, n\}$.

For $1 \leq j \leq n$ and $1 \leq i \leq r$, let $\epsilon_{i,j}$ be the idempotent in the group algebra of G_n given by

$$(6.13) \quad \epsilon_{i,j} := \frac{1}{r^j \cdot j!} \sum_{g \in \mathbb{Z}_r^l \mathfrak{S}_j} \text{sign}(g) \cdot \overline{\chi(g)}^i \cdot g \in \mathbb{C}[G_n].$$

(Recall that $\chi(g)$ is the product of the nonzero entries in the $j \times j$ monomial matrix g .) The idempotent $\epsilon_{i,j}$ commutes with the action of G_{n-j} . In particular, if V is a G_n -module, then $\epsilon_{i,j}V$ is a G_{n-j} -module. The relationship between $\text{Frob}(V)$ and $\text{Frob}(\epsilon_{i,j}V)$ is as follows.

Lemma 6.10. *Let V be a G_n -module, let $1 \leq j \leq n$, and let $1 \leq i \leq r$. We have*

$$(6.14) \quad \text{Frob}(\epsilon_{i,j}V) = e_j(\mathbf{x}^{(i*)})^\perp \text{Frob}(V).$$

In particular, if V is graded, we have

$$(6.15) \quad \text{grFrob}(\epsilon_{i,j}V; q) = e_j(\mathbf{x}^{(i*)})^\perp \text{grFrob}(V; q).$$

Proof. The proof is a standard application of Frobenius reciprocity and symmetric function theory (and can be found in [9] in the case $r = 1$).

It suffices to prove this lemma when V is irreducible, so let $V = \mathbf{S}^\lambda$ for some r -partition $\lambda \vdash_r n$. Consider the parabolic subgroup $G_{(n-j,j)} \subseteq G_n$. Irreducible representations of $G_{(n-j,j)}$ have the form $\mathbf{S}^\mu \otimes \mathbf{S}^\nu$ for $\mu \vdash_r n-j$ and $\nu \vdash_r j$. By Frobenius reciprocity, we have

$$\begin{aligned} (\text{multiplicity of } \mathbf{S}^\mu \otimes \mathbf{S}^\nu \text{ in } \text{Res}_{G_{(n-j,j)}}^{G_n} \mathbf{S}^\lambda) &= (\text{multiplicity of } \mathbf{S}^\lambda \text{ in } \text{Ind}_{G_{(n-j,j)}}^{G_n} \mathbf{S}^\mu \otimes \mathbf{S}^\nu) \\ &= (\text{coefficient of } s_\lambda(\mathbf{x}) \text{ in } s_\mu(\mathbf{x}) \cdot s_\nu(\mathbf{x})). \end{aligned}$$

The coefficient of $s_\lambda(\mathbf{x})$ in the Schur expansion of $s_\mu(\mathbf{x}) \cdot s_\nu(\mathbf{x})$ is

$$c_{\mu, \nu}^\lambda := c_{\mu^{(1)}, \nu^{(1)}}^{\lambda^{(1)}} \cdots c_{\mu^{(r)}, \nu^{(r)}}^{\lambda^{(r)}},$$

where the numbers $c_{\mu^{(1)}, \nu^{(1)}}^{\lambda^{(1)}}, \dots, c_{\mu^{(r)}, \nu^{(r)}}^{\lambda^{(r)}}$ are Littlewood-Richardson coefficients.

By the last paragraph, we have the isomorphism of $G_{(n-j,j)}$ -modules

$$(6.16) \quad \text{Res}_{G_{(n-j,j)}}^{G_n} \mathbf{S}^\lambda \cong \bigoplus_{\substack{\mu \vdash_r n-j \\ \nu \vdash_r j}} c_{\mu, \nu}^\lambda (\mathbf{S}^\mu \otimes \mathbf{S}^\nu),$$

which implies the isomorphism of G_{n-j} -modules

$$(6.17) \quad \epsilon_{i,j} \mathbf{S}^\lambda \cong \bigoplus_{\substack{\mu \vdash_r n-j \\ \nu \vdash_r j}} c_{\mu, \nu}^\lambda (\mathbf{S}^\mu \otimes \epsilon_{i,j} \mathbf{S}^\nu).$$

However, since the idempotent $\epsilon_{i,j}$ projects onto the $\nu_0 := (\emptyset, \dots, (1^j), \dots, \emptyset)$ -isotypic component of any G_j -module (where the nonempty partition is in position i), we have

$$(6.18) \quad \epsilon_{i,j} \mathbf{S}^\nu = \begin{cases} \mathbf{S}^{\nu_0} & \nu = \nu_0 \\ 0 & \nu \neq \nu_0. \end{cases}$$

Since \mathbf{S}^{ν_0} is 1-dimensional, we deduce

$$(6.19) \quad \epsilon_{i,j} \mathbf{S}^\lambda \cong \bigoplus_{\mu \vdash_r n-j} c_{\mu, \nu_0}^\lambda \mathbf{S}^\mu,$$

or

$$(6.20) \quad \text{Frob}(\epsilon_{i,j} S^\lambda) = \sum_{\mu \vdash_r n-j} c_{\mu, \nu_0}^\lambda s_\mu(\mathbf{x}).$$

To complete the proof, observe that $\text{Frob}(S^{\nu_0}) = e_j(\mathbf{x}^{(i)})$ and apply the definition of adjoint operators (together with the dualizing operation $i \mapsto i^*$ in the relevant inner product $\langle \cdot, \cdot \rangle$). \square

We will need to consider the action of the idempotent $\epsilon_{i,j}$ on polynomials in $\mathbb{C}[\mathbf{x}_n]$. Our basic tool is the following lemma describing the action of $\epsilon_{i,j}$ on monomials in the variables x_{n-j+1}, \dots, x_n .

Lemma 6.11. *Let (a_{n-j+1}, \dots, a_n) be a length j sequence of nonnegative integers and consider the corresponding monomial $x_{n-j+1}^{a_{n-j+1}} \cdots x_n^{a_n}$. Unless the numbers a_{n-j+1}, \dots, a_n are distinct and all congruent to $-i$ modulo r , we have*

$$(6.21) \quad \epsilon_{i,j} \cdot (x_{n-j+1}^{a_{n-j+1}} \cdots x_n^{a_n}) = 0.$$

Furthermore, if $(a'_{n-j+1}, \dots, a'_n)$ is a rearrangement of (a_{n-j+1}, \dots, a_n) , we have

$$(6.22) \quad \epsilon_{i,j} \cdot (x_{n-j+1}^{a_{n-j+1}} \cdots x_n^{a_n}) = \pm \epsilon_{i,j} \cdot (x_{n-j+1}^{a'_{n-j+1}} \cdots x_n^{a'_n}).$$

Proof. Recall that G_n acts on $\mathbb{C}[\mathbf{x}_n]$ by linear substitutions. In particular, if $1 \leq \ell \leq n$ and $\pi \in \mathfrak{S}_n \subseteq G_n$, we have $\pi.x_\ell = x_{\pi\ell}$. Moreover, if $g = \text{diag}(g_1, \dots, g_n) \in G_n$ is a diagonal matrix, we have $g.x_\ell = g_\ell^{-1}x_\ell$. Using these rules, the lemma is a routine computation. \square

The group G_j acts on the quotient ring $V_{n,k,j} := \mathbb{C}[x_{n-j+1}, \dots, x_n] / \langle x_{n-j+1}^{kr}, \dots, x_n^{kr} \rangle$. For any $1 \leq i \leq r$, let $\epsilon_{i,j}V_{n,k,j}$ be the image of $V_{n,k,j}$ under $\epsilon_{i,j}$. Then $\epsilon_{i,j}V_{n,k,j}$ is a graded vector space on which the idempotent $\epsilon_{i,j}$ acts as the identity operator. As a consequence of Lemma 6.11, the set of polynomials

$$(6.23) \quad \{\epsilon_{i,j} \cdot (x_{n-j+1}^{a_{n-j+1}} \cdots x_n^{a_n}) : 0 \leq a_{n-j+1} < \cdots < a_n < kr, a_\ell \equiv -i \pmod{r} \text{ for all } \ell\}$$

descends to a basis for $\epsilon_{i,j}V_{n,k,j}$. Counting the degrees of the monomials appearing in the above set, we have the Hilbert series

$$(6.24) \quad \text{Hilb}(\epsilon_{i,j}V_{n,k,j}; q) = q^{j \cdot (r-i) + r \cdot \binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{q^r}.$$

The following generalization of [14, Lem. 6.8] uses the spaces $\epsilon_{i,j}V_{n,k,j}$ to relate the modules $\epsilon_{i,j}S_{n,k}$ and $S_{n-j,k,k-j}$.

Lemma 6.12. *As graded G_j -modules we have $\epsilon_{i,j}S_{n,k} \cong S_{n-j,k,k-j} \otimes \epsilon_{i,j}V_{n,k,j}$.*

Proof. Write $\mathbf{y}_{n-j} = (y_1, \dots, y_{n-j}) = (x_1, \dots, x_{n-j})$ and $\mathbf{z}_j = (z_1, \dots, z_j) = (x_{n-j+1}, \dots, x_n)$, so that $\mathbb{C}[\mathbf{x}_n] = \mathbb{C}[\mathbf{y}_{n-j}, \mathbf{z}_j]$. The operator $\epsilon_{i,j} \in \mathbb{C}[G_j]$ acts on the \mathbf{z} variables and commutes with the \mathbf{y} variables.

There is a natural multiplication map

$$(6.25) \quad \tilde{\mu} : \mathbb{C}[\mathbf{y}_{n-j}] \otimes \epsilon_{i,j}V_{n,k,j} \rightarrow \epsilon_{i,j}\mathbb{C}[\mathbf{x}_n] / \epsilon_{i,j}J_{n,k} \cong \epsilon_{i,j}S_{n,k}$$

coming from the assignment $f(\mathbf{y}_{n-j}) \otimes g(\mathbf{z}_j) \mapsto f(\mathbf{y}_{n-j})g(\mathbf{z}_j)$. The map $\tilde{\mu}$ commutes with the action of G_{n-j} on the \mathbf{y} variables. We show that $\tilde{\mu}$ descends to the desired isomorphism.

We calculate

$$(6.26) \quad \epsilon_{i,j}(e_d(\mathbf{y}_{n-j}^r, \mathbf{z}_j^r)) = \sum_{a+b=d} e_a(\mathbf{y}_{n-j}^r) \epsilon_{i,j}(e_b(\mathbf{z}_j^r)) = e_d(\mathbf{y}_{n-j}^r)$$

for any $d > 0$. It follows that $e_d(\mathbf{y}_{n-j}^r) \in \epsilon_{i,j}J_{n,k}$ for all $d > n - k$. For any $f(\mathbf{z}_j) \in \epsilon_{i,j}V_{n,k,j}$ we have

$$(6.27) \quad \tilde{\mu}(y_\ell^{kr} \otimes f(\mathbf{z}_j)) = y_\ell^{kr} f(\mathbf{z}_j) = y_\ell^{kr} \epsilon_{i,j}(f(\mathbf{z}_j)) = \epsilon_{i,j}(y_\ell^{kr} f(\mathbf{z}_j)) \in \epsilon_{i,j}J_{n,k},$$

where we used the fact that $\epsilon_{i,j}$ acts as the identity operator on $\epsilon_{i,j}V_{n,k,j}$.

By the last paragraph, we have $J_{n-j,k,k-j} \otimes \epsilon_{i,j}V_{n,k,j} \subseteq \text{Ker}(\tilde{\mu})$. The map $\tilde{\mu}$ therefore induces a map

$$(6.28) \quad \mu : S_{n-j,k,k-j} \otimes \epsilon_{i,j}V_{n,k,j} \rightarrow \epsilon_{i,j}\mathbb{C}[\mathbf{x}_n]/\epsilon_{i,j}J_{n,k} \cong \epsilon_{i,j}S_{n,k}.$$

To determine the dimension of the target of μ , consider the action of $\epsilon_{i,j}$ on $\mathbb{C}[\mathcal{OP}_{n,k}]$. Given $\sigma \in \mathcal{OP}_{n,k}$, we have $\epsilon_{i,j}.\sigma = 0$ if and only if two of the big letters $n-j+1, \dots, n-1, n$ lie in the same block of σ . Moreover, if σ' is obtained from σ by rearranging the letters $n-j+1, \dots, n-1, n$ and/or changing their colors, then $\epsilon_{i,j}.\sigma'$ is a scalar multiple of $\epsilon_{i,j}.\sigma$. By Theorem 4.12, the dimension of the target of μ is

$$(6.29) \quad \dim(\epsilon_{i,j}S_{n,k}) = \dim(\epsilon_{i,j}\mathbb{C}[\mathcal{OP}_{n,k}]) = \binom{k}{j} \cdot |\mathcal{OP}_{n-j,k,k-j}|,$$

where the binomial coefficient $\binom{k}{j}$ comes from deciding which of the k blocks of σ receive the j big letters. On the other hand, Lemma 6.9 and the discussion after Lemma 6.11 imply that the domain of μ also has dimension given by (6.29). To prove that μ gives the desired isomorphism, it is therefore enough to show that μ is surjective.

To see that μ is surjective, let $\mathcal{C}_{n,k,j}$ be the set of polynomials of the form $\epsilon_{i,j}m(\mathbf{x}_n)$, where $m(\mathbf{x}_n) = m(\mathbf{y}_{n-j}) \cdot m(\mathbf{z}_j) \in \mathcal{N}_{n,k}$ has the property that $m(\mathbf{z}_j) = z_1^{a_1} \cdots z_j^{a_j}$ with $a_1 < \cdots < a_j$ and $a_\ell \equiv -i \pmod{r}$ for all ℓ . We claim that $\mathcal{C}_{n,k,j}$ descends to a basis of $\epsilon_{i,j}S_{n,k}$.

Since $\mathcal{N}_{n,k}^r$ is a basis of $S_{n,k}$, the set $\{\epsilon_{i,j}m(\mathbf{x}_n) : m(\mathbf{x}_n) \in \mathcal{N}_{n,k}\}$ spans $\epsilon_{i,j}S_{n,k}$. Let $m(\mathbf{x}_n) = m(\mathbf{y}_{n-j}) \cdot m(\mathbf{z}_j) \in \mathcal{N}_{n,k}$. By Lemma 6.11, we have $\epsilon_{i,j}m(\mathbf{x}_n) = 0$ unless $m(\mathbf{z}_j) = z_1^{a_1} \cdots z_j^{a_j}$ with (a_1, \dots, a_j) distinct and $a_\ell \equiv -i \pmod{r}$ for all ℓ . Also, if $m(\mathbf{z}_j)' = z_1^{a'_1} \cdots z_j^{a'_j}$ for any permutation (a'_1, \dots, a'_j) of (a_1, \dots, a_j) , then $\epsilon_{i,j}m(\mathbf{x}_n) = \pm \epsilon_{i,j}m(\mathbf{y}_{n-j}) \cdot m(\mathbf{z}_j)'$. It follows that $\mathcal{C}_{n,k,j}$ descends to a spanning set of $\epsilon_{i,j}S_{n,k}$.

Lemmas 6.7, 6.8, and 6.9 imply

$$(6.30) \quad |\mathcal{C}_{n,k,j}| = \binom{k}{j} \cdot |\mathcal{OP}_{n-j,k,k-j}| = \dim(\epsilon_{i,j}S_{n,k}).$$

It follows that $\mathcal{C}_{n,k,j}$ descends to a basis of $\epsilon_{i,j}S_{n,k}$.

Consider a typical element $\epsilon_{i,j}m(\mathbf{x}_n) = m(\mathbf{y}_{n-j}) \cdot \epsilon_{i,j}m(\mathbf{z}_j) \in \mathcal{C}_{n,k,j}$. We have

$$(6.31) \quad \mu(m(\mathbf{y}_{n-j}) \otimes \epsilon_{i,j}m(\mathbf{z}_j)) = m(\mathbf{y}_{n-j}) \cdot \epsilon_{i,j}m(\mathbf{z}_j) = \epsilon_{i,j}m(\mathbf{x}_n),$$

so that $\epsilon_{i,j}m(\mathbf{x}_n)$ lies in the image of μ . It follows that μ is surjective. \square

By Lemma 6.12, we have

$$(6.32) \quad e_j(\mathbf{x}^{(i^*)})^\perp \text{grFrob}(S_{n,k}; q) = \text{Hilb}(\epsilon_{i,j}V_{n,k,j}^r; q) \cdot \text{grFrob}(S_{n-j,k,k-r}^r; q)$$

$$(6.33) \quad = q^{j \cdot (r-i) + r \cdot \binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{q^r} \cdot \text{grFrob}(S_{n-j,k,k-r}^r; q).$$

If we want $\text{grFrob}(S_{n,k}; q)$ to satisfy the same recursion that $\mathbf{D}_{n,k}(\mathbf{x}; q)$ satisfies from Lemma 3.10, our goal is therefore

Lemma 6.13.

$$(6.34) \quad \text{grFrob}(S_{n-j,k,k-j}; q) = \sum_{m=\max(1,k-j)}^{\min(k,n-j)} q^{r \cdot (k-m) \cdot (n-j-m)} \begin{bmatrix} j \\ k-m \end{bmatrix}_{q^r} \text{grFrob}(S_{n-j,m}; q).$$

Proof. This is proven using the same reasoning as in the proofs of [14, Lem. 6.9, Lem. 6.10]; one just makes the change of variables $(x_1, \dots, x_n) \mapsto (x_1^r, \dots, x_n^r)$ and $q \mapsto q^r$. \square

We are ready to describe the graded isomorphism types of $S_{n,k}$ and $R_{n,k}$.

Theorem 6.14. *Let n, k , and r be positive integers with $n \geq k$ and $r \geq 2$. We have*

$$(6.35) \quad \text{grFrob}(S_{n,k}; q) = \mathbf{D}_{n,k}(\mathbf{x}; q)$$

and

$$(6.36) \quad \text{grFrob}(R_{n,k}; q) = \sum_{z=0}^{n-k} q^{krz} \cdot \mathbf{s}_{(\emptyset, \dots, \emptyset, (z))}(\mathbf{x}) \cdot \mathbf{D}_{n-z,k}(\mathbf{x}; q).$$

When $k = n$, the graded Frobenius image of $R_{n,n} = S_{n,n}$ was calculated by Stembridge [22].

Proof. By Lemma 6.13 (and the discussion preceding it), Lemma 3.10, and induction, we see that

$$(6.37) \quad e_j(\mathbf{x}^{(i^*)})^\perp \text{grFrob}(S_{n,k}; q) = e_j(\mathbf{x}^{(i^*)})^\perp \mathbf{D}_{n,k}(\mathbf{x}; q)$$

for all $j \geq 1$ and $1 \leq i \leq r$. Lemma 3.9 therefore gives the first statement. The second statement is a consequence of Proposition 6.1. \square

Example 6.15. Theorem 6.14 may be verified directly in the case $n = k = 1$. We have $S_{1,1} = R_{1,1} = \mathbb{C}[x_1]/\langle x_1^r \rangle$. The group $G_1 \cong G = \langle \zeta \rangle$ acts on $S_{1,1}$ by $\zeta \cdot x_1^i = \zeta^{-i} x_1^i$ for $0 \leq i < r$. Recalling our convention for the characters of the cyclic group G , we have

$$(6.38) \quad \text{grFrob}(S_{1,1}; q) = \mathbf{s}_{(\emptyset, \dots, \emptyset, (1))}(\mathbf{x}) \cdot q^0 + \dots + \mathbf{s}_{(\emptyset, (1), \dots, \emptyset)}(\mathbf{x}) \cdot q^{r-2} + \mathbf{s}_{((1), \emptyset, \dots, \emptyset)}(\mathbf{x}) \cdot q^{r-1}.$$

On the other hand, the elements of $\text{SYT}^r(1)$ are the tableaux

$$(\emptyset, \emptyset, \dots, \boxed{1}), \dots, (\emptyset, \boxed{1}, \dots, \emptyset), (\boxed{1}, \emptyset, \dots, \emptyset).$$

The major indices of these tableaux are (from left to right) $r-1, \dots, 1, 0$. By Proposition 3.8 we have

$$(6.39) \quad \mathbf{D}_{1,1}(\mathbf{x}; q) = \text{rev}_q [\mathbf{s}_{(\emptyset, \dots, \emptyset, (1))}(\mathbf{x}) \cdot q^{r-1} + \dots + \mathbf{s}_{(\emptyset, (1), \dots, \emptyset)}(\mathbf{x}) \cdot q^1 + \mathbf{s}_{((1), \emptyset, \dots, \emptyset)}(\mathbf{x}) \cdot q^0],$$

which agrees with Theorem 6.14.

Example 6.16. Let us consider Theorem 6.14 in the case $(n, k, r) = (3, 2, 2)$. By Proposition 3.8, the only elements of $\text{SYT}^2(3)$ which contribute to $\mathbf{D}_{3,2}(\mathbf{x}; q)$ are those with ≥ 1 descent.

$$\begin{array}{cccccccccccccccc} \boxed{1} & & \boxed{1} & \boxed{2} & & \boxed{1} & \boxed{3} & & \boxed{1} & & \boxed{1} & & \boxed{1} & & \boxed{1} & & \boxed{2} & & \boxed{2} \\ \boxed{2} & & \boxed{3} & & \boxed{2} & & \boxed{2} & & \boxed{2} & & \boxed{3} & & \boxed{3} & & \boxed{3} & & \boxed{3} & & \boxed{3} \\ \boxed{3} & & \emptyset & & \emptyset & & \emptyset & & \emptyset & & \emptyset & & \emptyset & & \emptyset & & \emptyset & & \emptyset \end{array}$$

The major indices of these tableaux are (in matrix format) $\begin{pmatrix} 6 & 4 & 2 & 7 & 5 & 3 & 3 & 5 \\ 8 & 4 & 6 & 6 & 4 & 7 & 5 & 9 \end{pmatrix}$ while the

descent numbers are $\begin{pmatrix} 2 & 1 & 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \end{pmatrix}$. The statistic $\text{maj}(\mathbf{T}) + r \binom{n-k}{2} - r(n-k)\text{des}(\mathbf{T})$

appearing in the exponent in Proposition 3.8 is therefore $\begin{pmatrix} 2 & 2 & 0 & 3 & 3 & 1 & 1 & 3 \\ 4 & 2 & 4 & 4 & 2 & 5 & 3 & 5 \end{pmatrix}$. If we apply ω and multiply by $[\text{des}(\mathbf{T})]_{q^r} = [\text{des}(\mathbf{T})]_{q^2}$, we see that $\mathbf{D}_{3,2}(\mathbf{x}; q)$ is the q -reversal of

$$(6.40) \quad \begin{aligned} & \mathbf{s}_{((3), \emptyset)}(\mathbf{x}) \cdot (q^2 + q^4) + \mathbf{s}_{((2,1), \emptyset)}(\mathbf{x}) \cdot q^2 + \mathbf{s}_{((2,1), \emptyset)}(\mathbf{x}) \cdot q^0 + \mathbf{s}_{((2), (1))}(\mathbf{x}) \cdot (q^3 + q^5) \\ & + \mathbf{s}_{((1,1), (1))}(\mathbf{x}) \cdot q^3 + \mathbf{s}_{((2), (1))}(\mathbf{x}) \cdot q^1 + \mathbf{s}_{((1,1), (1))}(\mathbf{x}) \cdot q^1 + \mathbf{s}_{((2), (1))}(\mathbf{x}) \cdot q^3 \\ & + \mathbf{s}_{((1), (2))}(\mathbf{x}) \cdot (q^4 + q^6) + \mathbf{s}_{((1), (1,1))}(\mathbf{x}) \cdot q^2 + \mathbf{s}_{((1), (2))}(\mathbf{x}) \cdot q^4 + \mathbf{s}_{((1), (1,1))}(\mathbf{x}) \cdot q^4 \\ & + \mathbf{s}_{((1), (2))}(\mathbf{x}) \cdot q^2 + \mathbf{s}_{(\emptyset, (2,1))}(\mathbf{x}) \cdot q^5 + \mathbf{s}_{(\emptyset, (2,1))}(\mathbf{x}) \cdot q^3 + \mathbf{s}_{(\emptyset, (3))}(\mathbf{x}) \cdot (q^5 + q^7). \end{aligned}$$

Collecting powers of q and applying rev_q , the graded Frobenius image $\text{grFrob}(S_{3,2}; q)$ is

$$(6.41) \quad \begin{aligned} & s_{(\emptyset, (3))}(\mathbf{x}) \cdot q^0 + s_{((1), (2))}(\mathbf{x}) \cdot q^1 + (s_{((2), (1))}(\mathbf{x}) + s_{(\emptyset, (2, 1))}(\mathbf{x}) + s_{(\emptyset, (3))}(\mathbf{x})) \cdot q^2 \\ & + (s_{((3), \emptyset)}(\mathbf{x}) + 2s_{((1), (2))}(\mathbf{x}) + s_{((1), (1, 1))}(\mathbf{x})) \cdot q^3 + (2s_{((2), (1))}(\mathbf{x}) + s_{((1, 1), (1))}(\mathbf{x}) + s_{(\emptyset, (2, 1))}(\mathbf{x})) \cdot q^4 \\ & + (s_{((3), \emptyset)}(\mathbf{x}) + s_{((2, 1), \emptyset)}(\mathbf{x}) + s_{((1), (1, 1))}(\mathbf{x}) + s_{((1), (2))}(\mathbf{x})) \cdot q^5 + (s_{((2), (1))}(\mathbf{x}) + s_{((1, 1), (1))}(\mathbf{x})) \cdot q^6 + s_{((2, 1), \emptyset)}(\mathbf{x}) \cdot q^7. \end{aligned}$$

Let us calculate $\text{grFrob}(R_{3,2}; q)$. A shorter calculation (left to the reader) shows that $\mathbf{D}_{2,2}(\mathbf{x}; q)$ is given by

$$(6.42) \quad \begin{aligned} & s_{(\emptyset, (2))}(\mathbf{x}) \cdot q^0 + s_{((1), (1))}(\mathbf{x}) \cdot q^1 + (s_{((2), \emptyset)}(\mathbf{x}) + s_{(\emptyset, (1, 1))}(\mathbf{x})) \cdot q^2 + s_{((1), (1))}(\mathbf{x}) \cdot q^3 + s_{((1, 1), \emptyset)}(\mathbf{x}) \cdot q^4. \end{aligned}$$

By Theorem 6.14, the Frobenius image $\text{grFrob}(R_{3,2}; q)$ is given by adding the product of (6.42) and $s_{(\emptyset, (1))}(\mathbf{x}) \cdot q^4$ to (6.41). Applying the Pieri rule we see that the Schur expansion of $\text{grFrob}(R_{3,2}; q)$ is

$$(6.43) \quad \begin{aligned} & (\text{expression in (6.41)}) + (s_{(\emptyset, (3))}(\mathbf{x}) + s_{(\emptyset, (2, 1))}(\mathbf{x})) \cdot q^4 + (s_{((1), (2))}(\mathbf{x}) + s_{((1), (1, 1))}(\mathbf{x})) \cdot q^5 \\ & + (s_{((2), (1))}(\mathbf{x}) + s_{(\emptyset, (2, 1))}(\mathbf{x}) + s_{(\emptyset, (1, 1, 1))}(\mathbf{x})) \cdot q^6 + (s_{((1), (2))}(\mathbf{x}) + s_{((1), (1, 1))}(\mathbf{x})) \cdot q^7 + s_{((1, 1), (1))}(\mathbf{x}) \cdot q^8. \end{aligned}$$

7. CONCLUSION

In this paper we introduced a quotient $R_{n,k}$ of the polynomial ring $\mathbb{C}[\mathbf{x}_n]$ whose structure is governed by the combinatorics of the set of k -dimensional faces $\mathcal{F}_{n,k}$ in the Coxeter complex attached to G_n , where $G_n = \mathbb{Z}_r \wr \mathfrak{S}_n$ is a wreath product.

Problem 7.1. *Let $W \subset GL_n(\mathbb{C})$ be a complex reflection group and let $0 \leq k \leq n$. Find a graded W -module $R_{W,k}$ which generalizes $R_{n,k}$.*

The quotient $R_{W,k}$ in Problem 7.1 should have combinatorics governed by the k -dimensional faces $\mathcal{F}_{W,k}$ of some Coxeter complex-like object attached to W . A natural collection of groups W to look at is the $G(r, p, n)$ family of reflection groups. Recall that, for positive integers r, p, n with $p \mid r$, the group $G(r, p, n)$ is defined by

$$(7.1) \quad G(r, p, n) := \{g \in G_n : \text{the product of the nonzero entries in } g \text{ is a } (r/p)^{\text{th}} \text{ root of unity}\}.$$

It is well known that the $G(r, p, n)$ -invariant polynomials $\mathbb{C}[\mathbf{x}_n]^{G(r, p, n)}$ have algebraically independent generators $e_1(\mathbf{x}_n^r), e_2(\mathbf{x}_n^r), \dots, e_{n-1}(\mathbf{x}_n^r)$, and $(x_1 \cdots x_n)^{r/p}$. However, even in the case of $G(2, 2, n)$ which is isomorphic to the real reflection group of type D_n , the authors have been unable to construct a quotient of $\mathbb{C}[\mathbf{x}_n]$ which carries an action of $G(2, 2, n)$ whose dimension is given by the number of k -dimensional faces in the D_n -Coxeter complex.

If W is any real reflection group and \mathbb{F} is any field, there is an \mathbb{F} -algebra $H_W(0)$ of dimension $|W|$ called the 0 -Hecke algebra attached to W . When W is the symmetric group \mathfrak{S}_n , there is an action of $H_W(0)$ on the polynomial ring $\mathbb{F}[\mathbf{x}_n]$ given by the isobaric Demazure operators (see [15]). When $W = \mathfrak{S}_n$, Huang and Rhoades proved that the ideal

$$(7.2) \quad \langle h_k(x_1), h_k(x_1, x_2), \dots, h_k(x_1, x_2, \dots, x_n), e_n(\mathbf{x}_n), e_{n-1}(\mathbf{x}_n), \dots, e_{n-k+1}(\mathbf{x}_n) \rangle \subseteq \mathbb{F}[\mathbf{x}_n]$$

is stable under this action, and that the corresponding quotient of $\mathbb{F}[\mathbf{x}_n]$ gives a graded version of a natural action of $H_{\mathfrak{S}_n}(0)$ on k -block ordered set partitions of $[n]$. This suggests the following problem.

Problem 7.2. *Let W be a real reflection group of rank n , let $H_W(0)$ be the 0 -Hecke algebra attached to W , and let $0 \leq k \leq n$. Describe a natural action of W on the set of k -dimensional faces in the Coxeter complex of W . Give a graded this action as a W -stable quotient of $\mathbb{F}[\mathbf{x}_n]$.*

Another possible direction for future research is motivated by the Delta Conjecture and the *Parking Conjecture* of Armstrong, Reiner, and Rhoades [2]. Let W be an irreducible real reflection group with reflection representation V and Coxeter number h , and consider a homogeneous system of parameters $\theta_1, \dots, \theta_n \in \mathbb{C}[V]_{h+1}$ of degree $h+1$ carrying the dual V^* of the reflection representation. Armstrong et. al. introduce an inhomogeneous deformation $(\Theta - \mathbf{x})$ of the ideal $(\Theta) = (\theta_1, \dots, \theta_n) \subseteq \mathbb{C}[V]$ generated by the θ_i and conjecture a relationship between the quotient $\mathbb{C}[V]/(\Theta - \mathbf{x})$ and $(W \times \mathbb{Z}_h)$ -set Park_W^{NC} of ‘ W -noncrossing parking functions’ defined via Coxeter-Catalan theory.

When $W = \mathfrak{S}_n$ is the symmetric group, the ‘classical’ h.s.o.p. quotient $\mathbb{C}[V]/(\Theta)$ is known to have graded Frobenius image given by (the image under ω of, after a q -shift) the Delta conjecture in the case $k = n$ at the specialization $t = 1/q$. In [14, Prob. 7.8] the problem was posed of finding a ‘ $k \leq n$ ’ extension of the Parking Conjecture for any real reflection group W . The authors are hopeful that the quotients studied in this paper will be helpful in this endeavor.

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REFERENCES

- [1] R. Adin, F. Brenti, and Y. Roichman. Descent representations and multivariate statistics. *Trans. Amer. Math. Soc.*, **357** (2005), 3051–3082.
- [2] D. Armstrong, V. Reiner, and B. Rhoades. Parking spaces. *Adv. Math.*, **269** (2015), 647–706.
- [3] E. Artin. *Galois Theory*, Second edition. Notre Dame Math Lectures, no. 2. Notre Dame: University of Notre Dame, 1944.
- [4] E. Bango and R. Biagoli. Colored-descent representations of complex reflection groups $G(r, p, n)$. *Israel J. Math.*, **160** (1) (2007), 317–347.
- [5] F. Bergeron. *Algebraic Combinatorics and Coinvariant Spaces*. CMS Treatises in Mathematics. Boca Raton: Taylor and Francis, 2009.
- [6] C. Chevalley. Invariants of finite groups generated by reflections. *Amer. J. Math.*, **77** (4) (1955), 778–782.
- [7] T. A. Dowling. A class of geometric lattices based on finite groups. *J. Combin. Theory Ser. B*, **14** (1973), 61–86.
- [8] A. M. Garsia. Combinatorial methods in the theory of Cohen-Macaulay rings. *Adv. Math.*, **38** (1980), 229–266.
- [9] A. M. Garsia and C. Procesi. On certain graded S_n -modules and the q -Kostka polynomials. *Adv. Math.*, **94** (1) (1992), 82–138.
- [10] A. M. Garsia and D. Stanton. Group actions on Stanley-Reisner rings and invariants of permutation groups. *Adv. Math.*, **51** (2) (1984), 107–201.
- [11] J. Haglund, M. Haiman, and N. Loehr. A combinatorial formula for nonsymmetric Macdonald polynomials. *Amer. J. of Math.*, **130** (2008), 359–383.
- [12] J. Haglund, N. Loehr, and J. Remmel. Statistics on wreath products, perfect matchings, and signed words. *European J. Combin.*, **26** (2005), 835–868.
- [13] J. Haglund, J. Remmel, and A. T. Wilson. The Delta Conjecture. Accepted, *Trans. Amer. Math. Soc.*, 2016. [arXiv:1509.07058](#).
- [14] J. Haglund, B. Rhoades, and M. Shimozono. Ordered set partitions, generalized coinvariant algebras, and the Delta conjecture. Preprint, 2016.
- [15] J. Huang and B. Rhoades. Ordered set partitions and the 0-Hecke algebra. Preprint, 2016. [arXiv:1611.01251](#).
- [16] I. G. Macdonald. *Symmetric Functions and Hall Polynomials*, Second edition. Oxford Mathematican Monographs. New York: The Clarendon Press Oxford University Press, 1995. With contributions by A. Zelevinsky, Oxford Science Publications.
- [17] J. Remmel and A. T. Wilson. An extension of MacMahon’s Equidistribution Theorem to ordered set partitions. *J. Combin. Theory Ser. A*, **134** (2015), 242–277.
- [18] B. Rhoades. Ordered set partition statistics and the Delta Conjecture. Preprint, 2016. [arXiv:1605.04007](#).
- [19] W. Specht. Eine Verallgemeinerung der symmetrischen Gruppe. *Schriften Math. Seminar (Berlin)*, **1** (1932), 1–32.
- [20] R. P. Stanley. Invariants of finite groups and their applications to combinatorics. *Bull. Amer. Math. Soc.*, **1** (1979), 475–511.
- [21] E. Steingrímsson. Statistics on Ordered Partitions of Sets. Preprint, 2014. [arXiv:0605670](#).

- [22] J. Stembridge. On the eigenvalues of reflection groups and wreath products. *Pacific J. Math.*, **140** (1989), 353–396.
- [23] B. Sturmfels. *Algorithms in Invariant Theory*. Springer-Verlag, Berlin, 1993.
- [24] A. T. Wilson. An extension of MacMahon’s Equidistribution Theorem to ordered multiset partitions. *Electron. J. Combin.*, **23** (1) (2016), P1.5.

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